QFI Quantitative Finance Exam

Exam QFIQF

Date: Thursday, April 27, 2023

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 11 questions numbered 1 through 11 with a total of 70 points.

   The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

3. Each question part or subpart should be answered either in the Excel document, or in the provided Yellow Answer Booklet. Graders will only look at the work in the Yellow Answer Booklet or Excel document.

4. In the Excel document, answers should be entered in the box marked ANSWER. The box will expand as lines of text are added. There is no need to use special characters or subscripts (though they may be used). For example, β_1 can be typed as beta_1 (and ^ used to indicate a superscript).

5. Prior to uploading your Excel files, each file should be saved and renamed with your five-digit candidate number in the filename.

6. The Excel file that contain your answers must be uploaded before the five-minute upload period expires.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

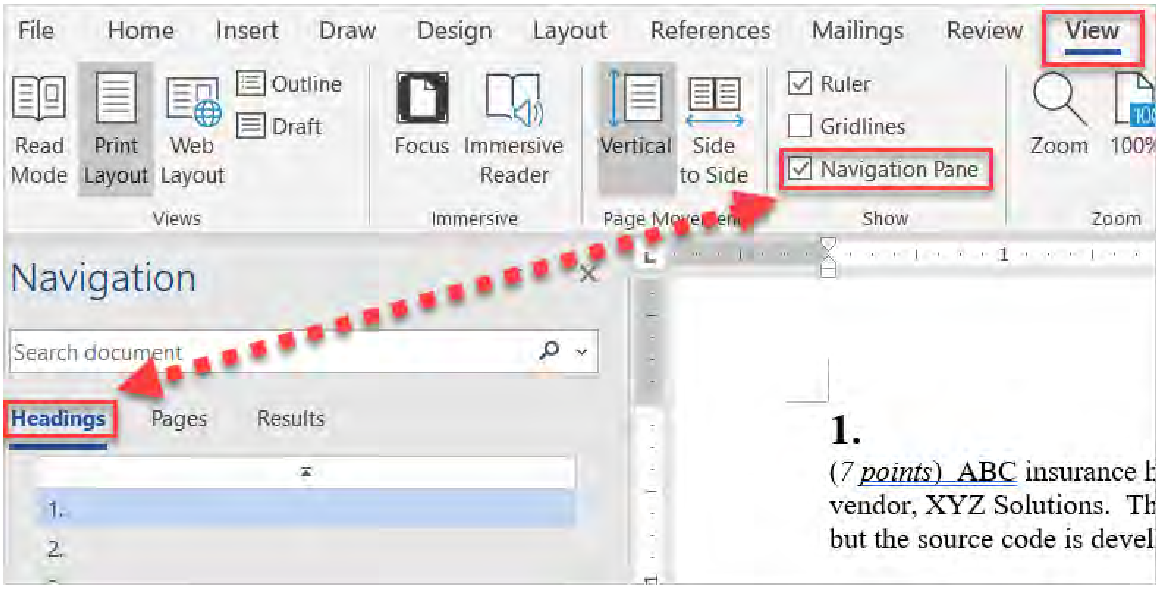
4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, hand in all your written-answer sheets to the Prometric center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

Navigation Instructions

Open the Navigation Pane to jump to questions.

Press Ctrl+F, or click View > Navigation Pane:



The responses for all parts of this question are required on the paper provided to you.

1.

(*5 points*) Let S_t denote the price of a financial asset at time t. Prices are observed at times t_i with $0 = t_0 < t_1 < \cdots < t_k = T$. At each time $t_i, i = 1,2,\cdots,k$, the price of the asset can increase or decrease by 1, with probability distribution:

$$P\left(\Delta S_{t_i} = 1\right) = p, \qquad \text{and} \qquad P\left(\Delta S_{t_i} = -1\right) = 1 - p,$$

where $0 < p < 1$ and $\Delta S_{t_i} = S_{t_i} - S_{t_{i-1}}$.

The price changes corresponding to different time intervals are independent.

Let $1 < m < k$.

(a) (*1 point*) Calculate $P(S_T - S_0 = 2m - k)$.

(b) (*1 point*) Determine p such that $\{S_{t_i}\}$ is a martingale with respect to the information set generated by the past price changes and the probability distribution P.

Assume $S_0 = 1$, $k = 4$, $p = 0.75$, and let I_i be the information set generated by the price changes up to time t_i. In addition, let $r = 0.05$ be the risk-free effective rate in (t_0, t_1) and Q be the risk-neutral measure.

(c) (*1 point*) Express $E^P[S_T|I_1]$ and $E^P[S_T|I_3]$ in terms of S_{t_1} and S_{t_3}, respectively.

(d) (*1 point*) Compute $Q\left(\Delta S_{t_1} = 1\right)$ and $Q\left(\Delta S_{t_1} = -1\right)$.

(e) (*1 point*) Explain the following tools used for asset pricing:

(i) The martingale representation theorem

(ii) Normalization

(iii) Change of measure

The responses for all parts of this question are required on the paper provided to you.

2.

(*7 points*) Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space and let $W(t)$ be a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Suppose that the risk-free rate is 0% and that $M(t)$ is a risky asset such that:

$$M(t) = M(0)e^{\alpha W(t) - \frac{\alpha^2 t}{2}},\ 0 < M(0) < \infty, \text{ and } \alpha \text{ is a constant.}$$

(a) (*2.5 points*) Show that $M(t)$ is a $\mathbb{Q}$-martingale using each of the following approaches:

(i) Deriving the stochastic dynamics of $M(t)$.

(ii) Applying the definition of a martingale.

Consider another risky asset $A(t) = A(0)e^{\vartheta W(t) - \frac{\vartheta^2 t}{2}}$, $\quad 0 < A(0) < \infty$ and ϑ is a constant and $\vartheta \neq \alpha$.

Define a new measure $\mathbb{Q}^A$ so that A is the numeraire asset under $\mathbb{Q}^A$.

(b) (*0.5 points*) Write down the Radon-Nikodym derivative of $\mathbb{Q}^A$ with respect to $\mathbb{Q}$.

(c) (*1.5 points*) Determine, using Ito's lemma and Girsanov Theorem, whether the normalized process $\frac{M(t)}{A(t)}$ is a $\mathbb{Q}$-martingale or a $\mathbb{Q}^A$-martingale.

(d) (*2.5 points*) Derive an expression for today's price of an exchange option with payoff $P(T) = max[0, M(T) - A(T)]$.

The responses for all parts of this question are required on the paper provided to you.

3.

(*4 points*) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{W_t\}_{t\geq 0}$ and $\{V_t\}_{t\geq 0}$ be standard Brownian motions with respect to the filtration $\mathcal{F}_t$. Let S_t and E_t denote a Canadian asset price quoted in USD and the USD-to-CAD exchange rate, respectively, with the following dynamics:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma_S S_t dW_t \\
dE_t &= \sigma_E E_t dV_t \\
dW_t dV_t &= \rho dt
\end{aligned}
$$

where $\mu, \sigma_S, \sigma_E, \rho$ are constant and $\rho > -1$.

(a) (*1.5 points*) Show that

$$Z_t = \frac{\sigma_S W_t + \sigma_E V_t}{\sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}}$$

is a $\mathbb{P}$-standard Brownian motion.

Let r be the Canadian risk-free rate.

(b) (*1 point*) Determine whether $\ln(S_t E_t)$ follows an arithmetic Brownian motion under the measure $\mathbb{P}$ or not.

(c) (*1.5 points*) Show that $e^{-rt} S_t E_t$ is a martingale under the risk-neutral measure $\mathbb{Q}$ using Girsanov Theorem, with the numeraire being CAD risk-free assets.

The responses for all parts of this question are required on the paper provided to you.

4.

(*6 points*) You are given that W_t is a standard Wiener process.

Consider the process $F_t = f(t)g(W_t)$, with f and g differentiable and $0 \le t < T$.

(a) (*1 point*) Prove by using Ito's lemma that

$$\int_0^T f(t)g'(W_t)\, dW_t$$
$$= f(t)g(W_t)|_0^T - \int_0^T f'(t)g(W_t)dt - \frac{1}{2}\int_0^T f(t)g''(W_t)\, dt.$$

Consider the following Ito integrals,

$I_T = \int_0^T t dW_t$ and $Y_T = \int_0^T W_t dt$.

You are given that $Var(I_T) = Var(Y_T) = \frac{T^3}{3}$.

(b) (*2 points*) Determine $Corr(I_T, Y_T)$.

(c) (*1.5 points*) Show that

$$\int_0^T \frac{2W_t}{1+W_t^2}\, dW_t = \ln(1 + W_T^2) - \int_0^T \frac{1-W_t^2}{(1+W_t^2)^2}\, dt.$$

You are given that for all x the following inequality holds:

$$\frac{1 - x^2}{(1 + x^2)^2} \le 1.$$

(d) (*0.5 points*) Show that $\int_0^T \frac{1-W_t^2}{(1+W_t^2)^2} dt \le T$.

(e) (*1 point*) Show that $E[\ln(1 + W_T^2)] \le T$.

The responses for all parts of this question are required on the paper provided to you.

5.

(*8 points*) Using the current yield curve and cap prices you have calibrated the Hull-White model below

$$dr_t = (\theta_t - \gamma r_t)dt + \sigma dX_t.$$

The data set contains 20 treasury bond prices and 5 cap prices. You fitted the following curve for $r(0,t)$ (continuously compounded yield between 0 and t)

$$r(0,t) = 0.01091858598 + 0.01251008594\, t - 0.000140114635\, t^2 + 0.005654825\, t^3$$

and estimated values of γ, σ to be 0.19, 0.0196 respectively.

(a) (*1 point*) Explain whether the fitted model is a true arbitrage-free model.

(b) (*1 point*) Derive an expression for the instantaneous forward rate at time 0 $f(0,t)$.

(c) (*1 point*) Derive an expression for θ_t .

(d) (*2 points*) Compute $E[r_{1.25}|r_1 = 0.03\%]$, given $f(0,1.25) = 0.036068$.

You are given the following for part (e):

- $r(0,4) = 0.04215968$
- Forward bond price volatility of a current 4 year zero-coupon bond one year from now, $S_Z(1,4)$, is 0.04088.

(e) (*3 points*) Compute the price of a European call option on a zero-coupon bond with the following specifications:

- The underlying bond is a 4-year zero-coupon bond at issue of the option.
- The option matures in one year.
- The strike price is 80 out of 100 principal.

6.

(*7 points*) Consider the stochastic differential equation for the spot rate r_t

$$dr_t = \alpha(m - r_t)dt + \sigma dW_t$$

where α, m, σ are constant, and W_t is a standard Wiener process.

(a) (*1 point*) Show that for $s < t$

(i) $E[r_t|\mathcal{F}_s] = r_s e^{-\alpha(t-s)} + m\left(1 - e^{-\alpha(t-s)}\right)$,

The response for this part is required on the paper provided to you.

(ii) $Var[r_t|\mathcal{F}_s] = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-s)}\right)$.

The response for this part is required on the paper provided to you.

Let $B(t,T) = E_t\left[e^{-\int_t^T r_u du}\right]$ be the price at time t of the zero-coupon bond maturing at time T.

You are given that $\int_t^T \int_t^u f(s,u)\, dW_s du = \int_t^T \int_s^T f(s,u) du\, dW_s$ for any differentiable function $f(s,u)$.

(b) (*1.5 points*) Show that $B(t,T)$ follows a log-normal distribution.

The response for this part is required on the paper provided to you.

(c) (*0.5 points*) Show that callable bond price at the call maturity is $min(\text{non callable Bond Price, Strike Price of the call})$ when callable bond is a combination of a straight bond and a call option on the bond.

The response for this part is required on the paper provided to you.

The risk-neutral parameters are $m = 0.0634$, $\alpha = 0.4653$, with $r_0 = 0.0162, \sigma = 0.0221$.

You are also given the following:

	Straight Bond	Callable Bond	Call Option
Notional (of Underlying Bond)	100	100	100
Coupon (of Underlying Bond)	5% per annum paid semiannually	5% per annum paid semiannually	5% per annum paid semiannually
Maturity T_B (of Underlying Bond)	2 years	2 years	2 years
Exercise Date T_O	N/A	1 year	1 year
Call / Strike Price (K)	N/A	100	100

The interest rate r_K^*, which makes the value at T_O of the underlying bond of the call option equal to the strike price, is 4.60%.

(d) (*3 points*) Calculate

(i) the bond option price at time 0.

The response for this part is to be provided in the Excel spreadsheet.

(ii) the callable bond price using the embedded call option price and the straight bond price

The response for this part is to be provided in the Excel spreadsheet.

(e) (*1 point*)

(i) Describe a shortcoming of a one-factor Vasicek model.

The response for this part is required on the paper provided to you.

(ii) Explain how two-factor Vasicek model can resolve it.

The response for this part is required on the paper provided to you.

The responses for all parts of this question are required on the paper provided to you.

7.

(*6 points*) You have been asked to consider the following interest rate model for valuing a zero-coupon bond.

The Cox, Ingersoll and Ross (CIR) model for the short-term interest rate r_t:

$dr_t = \gamma^*(\overline{r^*} - r_t)dt + \sqrt{\alpha r_t}dX_t$ where γ^*, $\overline{r^*}$ and α are positive constant

with $\gamma^*\overline{r^*} > \frac{1}{2}\alpha$, and X_t is a Standard Brownian motion.

Let $Y_t = \ln(r_t)$.

(a) (*2 points*)

(i) (*1.5 points*) Show by using Ito's lemma that

$$dY_t = \left[\left(\gamma^*\overline{r^*} - \frac{1}{2}\alpha\right)e^{-Y_t} - \gamma^*\right]dt + \sqrt{\alpha}e^{-\frac{Y_t}{2}}dX_t.$$

(ii) (*0.5 points*) Explain why the drift term of dY_t is positive if Y_t gets too far below from 0.

Let $Z(r_t, t;\ T)$ be the price of a zero-coupon bond in the CIR model at time *t* with maturity *T*, with final payoff $Z(r_T, T;\ T) = 1$, satisfying the fundamental pricing equation.

The price of a zero-coupon bond is given by $Z(r_t, t; T) = e^{A(t;T) - B(t;T)r_t}$,

where $A(t;T)$ and $B(t;T)$ are functions of *t* and *T* shown in the given formula sheets.

(b) (*1.5 points*)

(i) Prove that $\lim_{t\to\infty} E[r_t\ |r_0] = \bar{r}^*\ and \lim_{t\to\infty} Var[r_t\ |r_0] = \frac{\bar{r}^*\alpha}{2\gamma}$.

(ii) Identify the distribution of $ln\ Z(r_t, t;\ T)$.

7. Continued

You are given: $B(0;10) = 4.256073,\ \psi_1 = \sqrt{\gamma^{*2} + 2\alpha} = 0.212132,$
$r_0 = 0.04, \bar{r}^* = 0.041$

(c) (*2.5 points*)

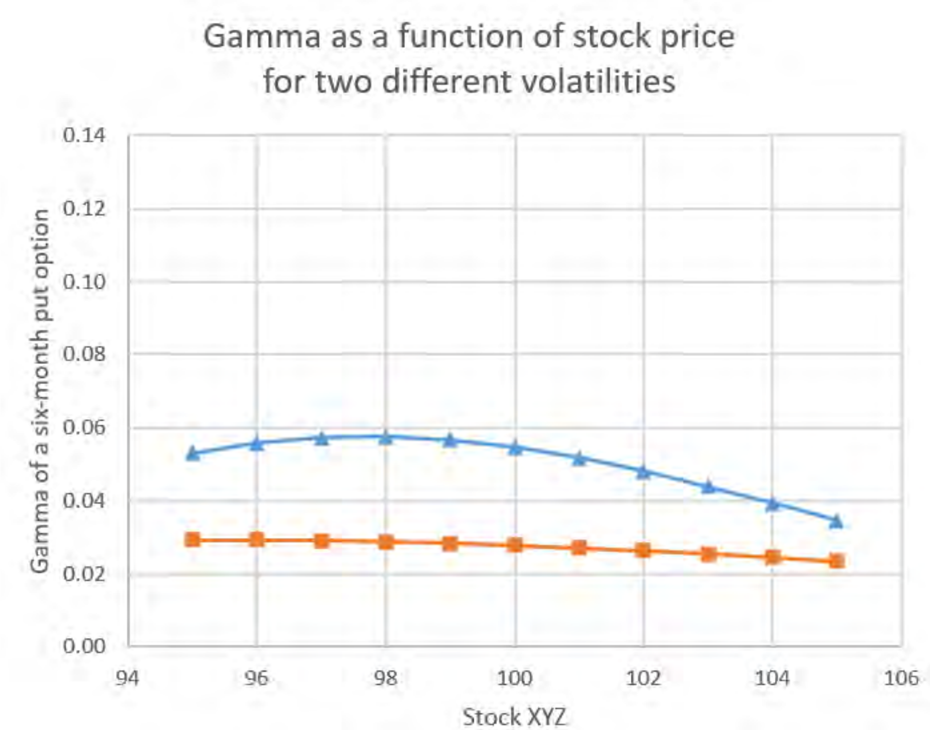
(i) Calculate γ^*, α.

(ii) Calculate $A(0;10),\ Z(r_0, 0; 10)$.

8.

(*7 points*) Consider a European put option on Stock XYZ, a non-dividend-paying stock. Also assume that:

- The spot price of Stock XYZ is 100 and the option strike is 100.
- The continuously compounded risk-free interest rate is 3%.



For part (a), use the chart above showing the evolution of option Gamma over a range of stock prices, also assume that:

- The time-to-maturity is six months.
- One curve is the option Gamma under 10% implied volatility while the other one is the option Gamma under 20% implied volatility.

(a) (*2 points*) Draw the two Gamma curves in the Excel spreadsheet.

The response for this part is to be provided in the Excel spreadsheet.

8. Continued

For part (b) and part (c), use the chart above showing the evolution of option Vega over the time and assume that:

- The implied volatility is constant 20%.
- One curve is the option Vega with a 3-month time-to-maturity while the other curve is the option Vega with a 12-month time-to-maturity.

(b) (*1 point*) Determine which curve corresponds to which time-to-maturity. Justify your answer.

The response for this part is required on the paper provided to you.

(c) (*2 points*) Draw the two Vega curves in the Excel spreadsheet.

The response for this part is to be provided in the Excel spreadsheet.

A popular trading strategy to protect investment from the high risk in the global equity markets is to purchase put options.

(d) (*2 points*) Critique the following two claims from the perspectives of directional traders and non-directional traders, respectively.

8. **Continued**

(i) "When buy a put option, you cannot lose more than the option premium paid for it."

The response for this part is required on the paper provided to you.

(ii) "To make a profit from the put purchase, realized volatility needs to be higher than the implied volatility."

The response for this part is required on the paper provided to you.

The responses for all parts of this question are required on the paper provided to you.

9.

(*5 points*) You are given a local volatility model where the local volatility varies only with the stock price according to:

$$\sigma = \min(15\% - 1.5 * \frac{S - S_0}{S_0}, 2\%)$$

Assume the following:

- Current value of the stock S_0 = 105;
- Annual continuous risk-free interest rate r= 4%;
- Annual continuous dividend yield q =1%

(a) (*2 points*) Construct the first four levels of a binomial tree for volatility and stock price with $\Delta t = \frac{1}{52}$ years by using the Cox-Ross-Rubinstein approach to construct the central spine of the tree. Show calculation for the first two levels clearly.

(b) (*1.5 points*) Calculate the price of a European put option with strike K= 102 that expires after four time-steps.

You are given that approximated current implied volatility is 17.1%.

The time-t price of a European put option with dividend q (for Black-Scholes model) is

$$N(-d_2)\, K * e^{-r(T-t)} - N(-d_1)\, S_t e^{-q(T-t)}$$

where $d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$,

S_t = index price at time t, K = strike price, T = maturity of the option.

(c) (*0.5 points*) Calculate the Black-Scholes put option value.

(d) (*1 point*) Explain why the Black-Scholes option value is different from the option value calculated using the binominal tree.

The responses for all parts of this question are required on the paper provided to you.

10.

(7 *points*) For its proposed GMMB rider offering, QFI Life is assessing 3 volatility management strategies with investments referencing an equity (S).

GMMB	Guaranteed	100% deposit
	Term	2 Years
Reference Equity (S)	Current Price (S_0)	100
	Current Volatility ($\sigma_{0,0}$)	20%
	Dividend (d)	0%
Other Assumptions	Death Rate (q)	0%
	Lapse Rate (l)	0%

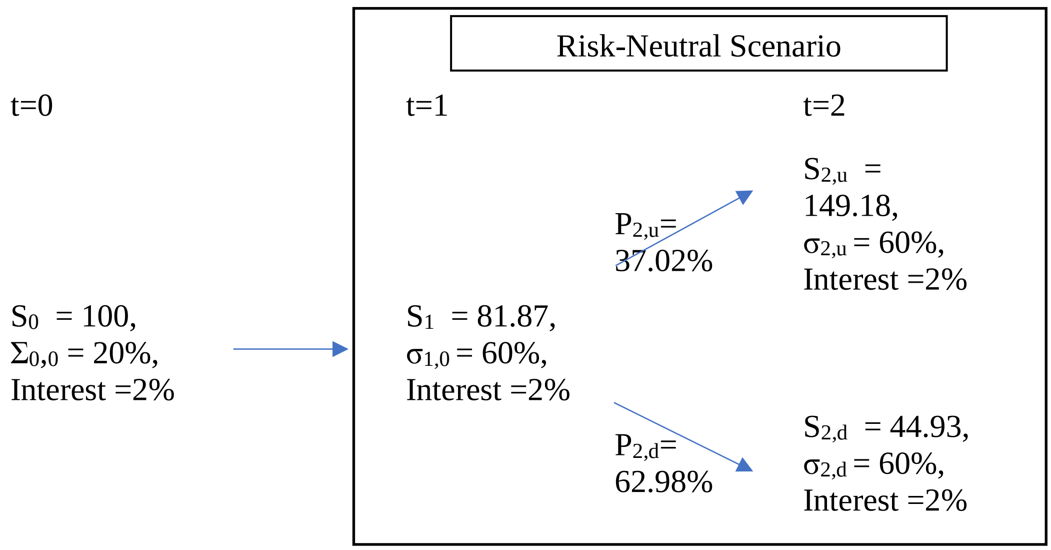
Volatility Management Strategies

1) Asset Transfer Program:
    - Guarantee Ratio (G%) = $1 - \frac{\textit{investment value}}{\textit{guaranteed value}}$
    - Asset Allocation to cash = Max [G%, 0%]
      (Note: allocation is to cash as opposed to fixed income)
    - Rider Fee = 0%
    - Rebalancing on an annual basis
2) Capped volatility fund:
    - Capped volatility at 60%
    - Equity ratio = min(100%, $\frac{60\%}{\textit{realized equity volatility}}$)
    - Rider Fee = 0%
    - Rebalancing on an annual basis
      (Note: Allocation is to cash as opposed to fixed income)
3) VIX-Indexed Fee:
    - Rider Fee = 0bps, if t=0; max[0bps, 200bps * ($\sigma_{t,j}$ -20%)], if t= 1, 2
    - $\sigma_{t,j}$ is the time-t volatility at stock level $S_{t,j}$ (where j =0, u, d)
    - Fee charged to investment value at the beginning of the year
    - Equity allocation is 100%
    - Fund Value = equity price (S) less rider fee

10. Continued

(a) (*1 point*) Describe the principal objectives for an insurer in designing an equity-based guarantee.

Your co-worker, Robin asks your help to build a prototype for risk-neutral scenario simulation. The risk-neutral scenario (from $t = 1$ to $t = 2$) is given below:



The interest rate is assumed to be continuously compounded and cash earns 0% interest rate. It is also assumed that the realized volatility turns out to be equal to the implied volatility, which is defined above.

(b) (*2 points*) Calculate the guarantee cost at the end of year 1 (t=1) for the GMMB rider under each of the 3 volatility management strategies. (Initial deposit = \$100)

Robin simulates 10,000 scenarios and analyzes the 2 key metrics for each of the volatility management strategies but forgets to label the results. (Initial deposit = $100)

Strategies	Guarantee Cost (t=0)	Hedge P&L (Cumulative)
100% Static allocation in Equity (S)	12.30	Loss of 4.6%
B	10.01	Loss of 1.2%
C	12.45	Loss of 5.0%
D	12.14	Loss of 1.5%
E	12.18	Loss of 4.1%

(c) (*2 points*) Identify the 4 volatility management strategies from the table above including no volatility management strategy.

Robin re-simulates the guaranteed cost under the two scenarios below:

Strategies	Guarantee Cost (t=0)	
	Volatility = 10%	Volatility = 40%
Asset Transfer Program	4.38	10.35
Capped volatility fund	7.55	20.91
VIX-Indexed Fee	7.35	21.50

(d) (*1 point*)

(i) Calculate the Vega under each of the 3 volatility management strategies (Hint: use finite difference approximation).

(ii) Explain how low Vega can benefit the hedge program.

(iii) Propose a volatility management strategy from the insurer's perspective based on the results in part (c) and (d) (i).

QFI's target market are high-net-worth clients, who are interested in, and willing to pay for upside investment potential. Your manager, Joe considers implementing the asset transfer program strategy.

(e) (*1 point*) Critique whether Joe's proposal meets the needs of the clients in the target market.

The responses for all parts of this question are required on the paper provided to you.

11.

(*8 points*) You are the actuary at XYZ in charge of constructing a hedging strategy for your company's VA product with a guaranteed minimum maturity benefit (GMMB) rider.

Your company uses a Heston model to simulate the underlying equity market dynamics, under the risk-neutral measure $\mathbb{Q}$:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^S$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma_v \sqrt{v_t} dW_t^v$$

where W_t^S and W_t^v are Wiener processes with correlation $\rho_v < 0$, and $r, \kappa, \theta, \sigma_v$ are constant with $2\kappa\theta > \sigma_v^2$.

You are calibrating the instantaneous variance process for hedging VA liabilities. You are considering the following approaches:

(i) Implied volatility surface computed from exchange traded vanilla options.

(ii) 1-year VIX index.

(iii) Historical realized volatility.

(a) (*1.5 points*) Explain the considerations when using each of the approaches above.

The rider fee (α) is currently assessed as a percentage of the policyholder's account value, such that the insurer's expected present value of prospective rider fees for the GMMB at time t is:

$$Y_t = E^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)}\alpha A_s ds\right]{}_{T-t}p_{x+t}$$
$$= A_t\left(1 - e^{-\alpha(T-t)}\right){}_{T-t}p_{x+t}$$

where ${}_{T-t}p_{x+t}$ = probability of a policyholder, who is *(x+t)* years old, surviving in the next *T-t* years.

You are given that the delta and vega of the GMMB liability net of fees are:

$$delta_t = \frac{\partial\Psi(t,T,A,G,v)}{\partial A}{}_{T-t}p_{x+t} - \left(1 - e^{-\alpha(T-t)}\right){}_{T-t}p_{x+t}$$

$$vega_t = \frac{\partial\Psi(t,T,A,G,v)}{\partial(\varsigma)}{}_{T-t}p_{x+t}$$

$where\ \varsigma = v^{\frac{1}{2}}$ and

$\Psi(t,T,A,G,v)$
$=$ time
$- t$ price of a put option with maturity T and strike price G , written on the account value A

Your manager has asked you to explore changing the rider fee as a percentage of the guaranteed payment.

(b) (*1 point*) Show that the insurer's expected present value of prospective rider fees becomes:

$$Y_t = \alpha G\left(\frac{1 - e^{-r(T-t)}}{r}\right){}_{T-t}p_{x+t}$$

(c) (*1 point*) Explain whether the following has increased, decreased, or remained the same after this change, from the insurer's perspective.

(i) Delta of the liability net of rider fees.

(ii) Vega of the liability net of rider fees.

11. Continued

You are asked to implement a new product feature, where the rider fee is an annual rate of the guaranteed payment paid continuously and indexed to the instantaneous variance:

$$\alpha_t = m + \lambda v_t$$

where m is a base fee, and $\lambda > 0$ is a scaling factor.

You are given for $t < s$:

$$E^{\mathbb{Q}}[v_s] = v_t e^{-\kappa(s-t)} + \theta\left(1 - e^{-\kappa(s-t)}\right)$$

(d) (*2.5 points*) Show that the fair value of prospective fees at time t, as defined as the risk-neutral expected present value of fees that will be collected by the insurer before the contract's maturity at time T, is:

$$L_t = G\left[(m + \lambda\theta)\left(\frac{1 - e^{-r(T-t)}}{r}\right) + \lambda(v_t - \theta)\left(\frac{1 - e^{-(r+\kappa)(T-t)}}{r + \kappa}\right)\right]{}_{T-t}p_{x+t}$$

(e) (*2 points*) Explain whether you agree or disagree with the following statements made by your analyst.

(i) "The new rider fee is not a function of A_t, therefore it is not sensitive to changes in the account value."

(ii) "The new rider fee has a positive Vega."

****END OF EXAMINATION****