Advanced Short-Term Actuarial Mathematics Exam

Exam ASTAM

Date: Wednesday, April 26, 2023

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 6 questions numbered 1 through 6 with a total of 60 points.

   The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

3. Each question part or subpart should be answered in the provided Yellow Answer Booklet. Graders will only look at the work in the Yellow Answer Booklet.

4. The Excel file will not be uploaded for grading, and therefore will NOT BE GRADED. It should be used for looking up values for statistical functions and may be used for calculations.

5. If you use Excel for calculations, you should include as much information in the Yellow Answer Booklet as if you had used a calculator, including formulas and intermediate calculations where relevant. Written answers without sufficient support will not receive full credit.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas in the Yellow Answer Booklet. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, hand in all your written-answer sheets to the Prometric center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

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BEGINNING OF EXAMINATION
ADVANCED SHORT-TERM ACTUARIAL MATHEMATICS

1.

(*12 points*) You are the actuary for an insurance company that writes one year insurance coverage on a risk with the following frequency and severity distributions.

Number of Losses	**Probability**
0	0.20
1	0.25
2	0.40
3	0.15

Amount of Loss	**Probability**
100	0.50
200	0.40
500	0.10

Let S denote the aggregate losses in a year for this risk.

(a) (*2 points*) Calculate the probability that $S \le 350$.

For each policy, the premium is set at 1.15 times the expected annual claim payments.

(b) (*1 point*) Calculate the premium for a policy with no policy modifications.

(c) (*2 points*) Show that the premium for a policy with an ordinary deductible of 100 per loss is 140 to the nearest 10. You should calculate the premium to the nearest 1.

(d) (*3 points*) Calculate the premium for a policy with an aggregate deductible of 100 per insured for each year.

(e) (*4 points*) The insurer plans to buy stop-loss reinsurance for a policy which has no coverage modifications.

(i) (*3 points*) Calculate the expected annual reinsurance payment with a stop-loss deductible of 1000.

(ii) (*1 point*) Explain why the insurer would prefer to use stop-loss coverage rather than setting a policy limit of 1000.

2.

(*10 points*) You are given the following claim frequency data for a line of business with 100 one-year policies.

Number of Claims (k)	0	1	2	3	4	5+
Policies (n_k)	46	34	13	4	3	0

You are particularly interested in the probability that a policy will have no claims in a year, denoted p_0.

You first fit a Poisson (λ) distribution to this data using Maximum Likelihood Estimation (MLE).

(a) (*3 points*) You are given that the MLE of λ is $\hat{\lambda} = 0.84$.

(i) (*1 point*) Show that the MLE of p_0 is $\hat{p}_0 = 0.43$ to the nearest 0.01. You should calculate the value to the nearest 0.001.

(ii) (*2 points*) Use the delta method to construct an approximate 95% linear confidence interval for p_0 under this model.

You next fit a zero-modified Poisson distribution to this data.

(b) (*3 points*)

(i) (*1 point*) Write down the likelihood function for the zero-modified Poisson distribution in terms of the model parameters (λ and p_0^M) and the data (n_k).

(ii) (*2 points*) Using the likelihood function, show that the MLE of p_0^M is $\hat{p}_0^M = 0.5$ to the nearest 0.1. You should calculate the value to the nearest 0.01.

(c) (*2 points*)

(i) Show that $\mathrm{Var}\left[\hat{p}_0^M\right] = \dfrac{p_0^M\left(1 - p_0^M\right)}{100}$.

(ii) Construct an approximate 95% linear confidence interval for p_0^M under this model.

2. **Continued**

You have calculated the following maximum log-likelihood values for the two models:

Model	Poisson	Zero-Modified Poisson
Maximum Log-likelihood	-124.36	-123.91

(d) (*2 points*)

(i) Calculate the Schwarz Bayes Criterion (SBC) for both models.

(ii) Your colleague claims that, because the SBC favors the Poisson model, the Poisson model will be superior for predicting the value of p_0. State with reasons whether or not your colleague is correct.

3.

(*11 points*) You are given that, under nonparametric empirical Bayes parameter estimation for the Bühlmann-Straub credibility model, X_{ij} denotes the claim frequency from risk group *i* in year *j*, and

$$\mathrm{Var}\left[X_{ij} \middle| \theta_i\right] = \frac{v(\theta_i)}{m_{ij}}.$$

(a) (*2 points*)

(i) Interpret the m_{ij} term.

(ii) Interpret θ_i.

An insurer issues auto insurance policies through two different brokers, Broker A and Broker B. Claim frequency data from the last three years is given in the following table. There were no policies issued through Broker B in Year 1.

Risk Group		Year 1	Year 2	Year 3	**Total**
Broker A	Number of Claims	39	28	15	**82**
	Number of Policies	200	150	180	**530**
Broker B	Number of Claims	n/a	16	23	**39**
	Number of Policies	n/a	50	60	**110**

Assume first that the insurer models the claim frequency for the two groups using Bühlmann-Straub credibility, with nonparametric empirical Bayes parameter estimation.

(b) (*3 points*)

(i) Show that $\hat{v}$ is 0.5 to the nearest 0.1. You should calculate the value to the nearest 0.001.

(ii) Show that $\hat{a}$ is 0.017 to the nearest 0.001. You should calculate the value to the nearest 0.0001.

(c) (*3 points*) Calculate the Bühlmann-Straub credibility estimate of the expected claim frequency for an individual policy issued through Broker B in Year 4.

3. **Continued**

(d) (*3 points*) Now assume that the insurer models the claim frequency for each group using a Poisson distribution. You are given the following information:

- The Poisson parameter for each broker is estimated by maximum likelihood, based on the data above.
- In year 4 it is assumed that for a randomly selected policy, there is a 75% probability that the policy was issued through Broker A, and a 25% probability that the policy was issued through Broker B.
- A randomly selected policy had two claims in Year 2 and two claims in Year 3.

(i) Show that the posterior probability that this policy was issued through Broker B is 0.86 to the nearest 0.01. You should calculate the value to the nearest 0.001.

(ii) Calculate the Bayesian estimate of the expected claim frequency for this policy in year 4.

4.

(*9 points*) A company is modeling the severity of flood losses. The following table shows the largest 20 values from a sample of 750 observations in decreasing order, with their ranks.

Rank	750	749	748	747	746	745	744	743	742	741
Loss	28,250	14,782	12,427	9,676	7,005	4,322	4,057	3,773	3,504	3,123

Rank	740	739	738	737	736	735	734	733	732	731
Loss	2,924	2,780	2,656	2,483	2,441	2,357	2,067	2,045	2,008	1,977

For convenience, you are also given that the total of the largest 10 losses is 90,919.

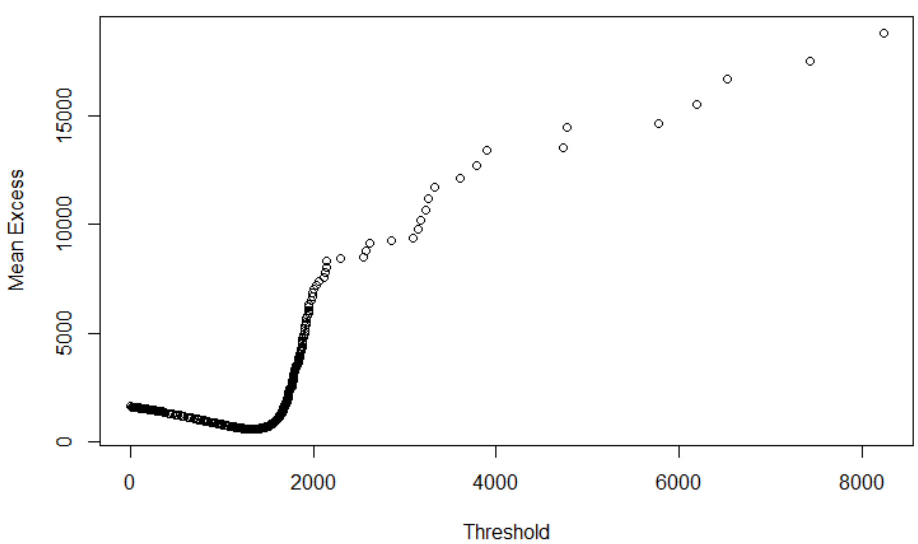
(a) (*3 points*)

(i) Estimate the 98.4% Value at Risk (VaR) of the losses, using the smoothed empirical method.

(ii) Estimate the 98.4% Expected Shortfall of the losses.

(iii) Calculate the empirical Mean Excess Loss, $\hat{e}(x)$, at $x = 3,000$.

The actuary decides to estimate the risk measures using the Generalized Pareto Distribution (GPD) for the excess loss distribution. She has created the following plot of the empirical mean excess loss function. (The last five points are excluded).



4. **Continued**

(b) (*1 point*) The actuary proposes a threshold of $d = 2{,}100$ for fitting the GPD. Her colleague points out that there is a gradient change at 1,500. He states that this would be a better threshold because it provides more data to fit the GPD.

Critique the colleague's suggestion.

(c) (*3 points*) The MLE estimators with $d = 2{,}100$ are $\hat{\xi} = 0.68$ and $\hat{\beta} = 2{,}000$.

(i) Show that the 98.4% VaR of the loss using the fitted GPD is 2,700 to the nearest 100. You should calculate the value to the nearest 1.

(ii) Calculate the 98.4% Expected Shortfall (ES) using the fitted GPD.

(d) (*2 points*)

(i) Explain why the VaR estimates in part (a) and in part (c) are similar, while the Expected Shortfall estimates are quite different.

(ii) Recommend which Expected Shortfall estimate the company should use for assessing its potential risk. Justify your answer.

5.

(*7 points*) You are conducting a ratemaking analysis for a line of business that is sold in only two geographical areas, Territory A and Territory B.

You are given the following information:

(i) The current base rate for Territory A and Age Group 1 is 200.

(ii) Age Group differentials will not change.

Territory	Age Group	Calendar Year 2022 Earned Exposures	Accident Year 2022 Losses as of December 31, 2022
A	1	5,000	600,000
A	2	2,200	242,000
B	1	3,100	434,000
B	2	1,300	169,000

Territory	Current Differential
A	1.00
B	1.20

Age Group	Current Differential
1	1.00
2	0.95

(a) (*2 points*) Show that the new differential for Territory B is 1.17 to the nearest 0.01, using the loss ratio method. You should calculate the value to the nearest 0.001.

(b) (*1 point*) State why losses do not need to be developed to ultimate values when calculating indicated differentials in part (a).

The company has decided to implement the following new territory differentials:

Territory	New Differential
A	1.00
B	1.15

(c) (*2 points*) Calculate the new Territory A, Age Group 1 base rate needed to give an overall rate increase of 7%.

5. **Continued**

(d) (*2 points*)

(i) Calculate the percentage increase in premiums in Territory B.

(ii) State with reasons whether you would expect the proportion of policies sold in Territory A to increase, decrease, or stay the same, as a result of the rate changes.

6.

(*11 points*)

(a) (*2 points*)

(i) Describe the purpose of Outstanding Claim Reserves (OCR).

(ii) Write down three reasons why there might be a delay between a loss event and the claim settlement.

You are given the following data on cumulative claims, $C_{i,j}$, for accident year (AY) i, $i = 0,1,...,4$, and development year (DY) j, $j = 0,1,...,4$.

Accident Year i	***Development Year j***				
	0	**1**	**2**	**3**	**4**
0	100	120	135	145	158
1	135	168	190	210	
2	175	210	225		
3	212	265			
4	260				

Let $X_{i,j}$ denote the incremental claims for AY i in DY j.

(b) (*2 points*) Use the chain ladder method to calculate the projected incremental claims $\hat{X}_{2,3}$ and $\hat{X}_{2,4}$.

(c) (*3 points*) The Poisson model for outstanding claims assumes that $X_{i,j}$ are independent random variables, with

$$X_{i,j} \sim \text{Poisson}\left(\mu_i \gamma_j\right), \text{ where } \mu_i > 0,\ \gamma_j > 0,\ \text{ and } \sum_{j=0}^{4} \gamma_j = 1 \ .$$

(i) Show that for any given i, j, $E[C_{i,j}] = \mu_i \sum_{k=0}^{j} \gamma_k$ under this model.

(ii) Interpret the μ_i parameters.

(iii) Interpret the γ_j parameters.

6. Continued

(d) (*3 points*) You are given that the chain ladder method generates the same estimate of projected claims as the Poisson model, with maximum likelihood estimation.

(i) Use the values in (b) to show that $\hat{\gamma}_3 = 0.0776$ and $\hat{\gamma}_4 = 0.0823$.

(ii) Calculate the estimated standard deviation of the expected outstanding claims for AY 2.

(iii) Using a normal approximation, calculate a 95% confidence interval for the expected outstanding claims for AY 2.

(e) (*1 point*) Your colleague notes that:

"*The chain ladder estimate gives the same outstanding claims estimate as the Poisson model, so that the chain ladder model is the same as the Poisson model*."

Critique your colleague's remarks.

****END OF EXAMINATION****