Advanced Long-Term Actuarial Mathematics

Solutions to Sample Questions

Most of the sample questions for the ALTAM exam are from past MLC or LTAM exams. The solutions given here are generally edited versions of the model solutions provided for the past exams. In some cases, we include comments from the graders, where that could be helpful for those learning the material. There may also be alternative solution methods that are not presented here.

Different solutions show different levels of accuracy in intermediate results. These model solutions are not intended to imply that this is the best rounding for each question. Graders do not penalize rounding decisions, unless an answer is rounded to too few digits in the context of the problem and the given information. In particular, if a problem in one step asks you to calculate something to the nearest 1, and you calculate it as (for example) 823.18, you need not bother saying "that's 823 to the nearest 1", and you may use 823.18 or 823 in future steps.

In the numerical solutions presented, there may be small rounding differences arising from the fact that values used in the calculations are typically more accurate than the intermediate values recorded.

Versions:

Feb. 26, 2023	Original set of 56 sample questions published for the ALTAM exam
Feb. 6, 2024	Added two sample Excel questions (Questions 57-58)
Jan. 20, 2025	Added one sample Excel question (Question 59) and two sample questions (Questions 60-61)
Mar. 9, 2025	Changed Question 60 to a sample Excel question and updated the solution of Questions 50 and 61
Apr. 10, 2025	Notation revision to Question 19 solution.

Current Version Dated April 10, 2025

Question 1

(a)

$$\frac{d}{dt}\,{}_tp_x^{10} = {}_tp_x^{11}\mu_{x+t}^{10} - {}_tp_x^{10}\left(\mu_{x+t}^{01} + \mu_{x+t}^{03}\right)$$

$$\frac{d}{dt}\,{}_tp_x^{11} = {}_tp_x^{10}\mu_{x+t}^{01} - {}_tp_x^{11}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12} + \mu_{x+t}^{13}\right)$$

Boundary Conditions: ${}_0p_x^{10} = 0 \qquad {}_0p_x^{11} = 1$

Graders' Comments:
Most candidates achieved full credit for this part. Common errors included omitting the boundary conditions, or omitting terms on the right hand side. A few candidates forgot the derivatives on the left hand side.

(b) From the Kolmogorov equation for $\frac{d}{dt}\,{}_tp_{80}^{10}$, we have

$$\lim_{h\to 0}\frac{{}_{t+h}p_{80}^{10} - {}_tp_{80}^{10}}{h} = {}_tp_{80}^{11}\mu_{80+t}^{10} - {}_tp_{80}^{10}\left(\mu_{80+t}^{01} + \mu_{80+t}^{03}\right)$$

So, for small h, $\dfrac{{}_{t+h}p_{80}^{10} - {}_tp_{80}^{10}}{h} \approx {}_tp_{80}^{11}\mu_{80+t}^{10} - {}_tp_{80}^{10}\left(\mu_{80+t}^{01} + \mu_{80+t}^{03}\right)$

$$\Rightarrow {}_{t+h}p_{80}^{10} \approx {}_tp_{80}^{10} + h\left[{}_tp_{80}^{11}\mu_{80+t}^{10} - {}_tp_{80}^{10}\left(\mu_{80+t}^{01} + \mu_{80+t}^{03}\right)\right]$$

$$\Rightarrow {}_{1/3}p_{80}^{10} \approx 0 + \frac{1}{3}(0.08) = 0.02667$$

and ${}_{2/3}p_{80}^{10} \approx {}_{1/3}p_{80}^{10} + \frac{1}{3}\left[(0.90346)(0.08) - (0.02667)(0.13082)\right] = 0.04960$

$${}_1p_{80}^{10} \approx {}_{2/3}p_{80}^{10} + \frac{1}{3}\left[(0.81652)(0.08) - (0.04960)(0.13186)\right] = 0.06919$$

Graders' Comments:
This part was quite well done. Many candidates calculated values for ${}_hp_{80}^{11}$, not realizing these had been given to them in the question.

(c) (i) The expected present value of the service fees is

$$EPV = 8000\left(\bar{a}_{80}^{00} + \bar{a}_{80}^{01}\right) = (8000)(5.5793 + 1.3813) = 55,685$$

(c) (ii) The expected present values of the level 2 care costs is

$$\begin{aligned} EPV &= (30{,}000)\bar{a}_{80}^{02} + (10{,}000)_{5|}\bar{a}_{80}^{02} \\ &= (30{,}000)\bar{a}_{80}^{02} + (10{,}000)(v^5\,{}_5p_{80}^{00}\;\bar{a}_{85}^{02} + v^5\;{}_5p_{80}^{01}\;\bar{a}_{85}^{12} + v^5\;{}_5p_{80}^{02}\;\bar{a}_{85}^{22}) = 23{,}005 \end{aligned}$$

Graders' Comments: *This part is designed to test understanding of the multiple state model annuity factor notation, and ability to manipulate the probabilities and annuities to create term annuity factors. It was one of the more challenging parts of the exam overall.*

Many candidates achieved full credit for (c)(i). Those who did not tended to combine, for example, $\bar{a}_{80}^{00}$ *and* $\bar{a}_{80}^{11}$*, which would require the person to be in state 0 and state 1 simultaneously at age 80.*

In (c)(ii), most candidates did not take all three cases into consideration. Some answers allowed appropriately for one or two cases, and these received partial credit. Some candidates combined, for example, ${}_5p_{80}^{02}$ *with* $\bar{a}_{80}^{02}$*, which is a more serious error, as it indicates a lack of understanding of the functions involved.*

Question 2

(a) Let $l_{x+t^-}^{(\tau)}$ denote the value immediately before exits at exact age $x+t$, and $l_{x+t^+}^{(\tau)}$ denote the value immediately after. At $t=0.5$, just before decrement 2 exits, we have

$$q_{60}'^{(2)} = \frac{d^{(2)}}{l_{60.5^-}^{(\tau)}}$$

$$l_{60.5^-}^{(\tau)} = 1000\exp\left(-\int_0^{0.5} 1.2t\ dt\right) = (1000)(0.86071) = 860.71$$

$$\Rightarrow q_{60}'^{(2)} = \frac{60}{860.71} = 0.0697$$

(b)

$$l_{60.5+}^{(\tau)} = 800.71 \Rightarrow l_{61^-}^{(\tau)} = 800.71e^{-\int_{0.5}^{1} 1.2t\ dt} = 800.71(0.63763) = 510.6$$

$$l_{61+}^{(\tau)} = 510.6 - 45 = 465.6$$

$$\Rightarrow d_{60}^{(1)} = l_{60}^{(\tau)} - d_{60}^{(2)} - d_{60}^{(3)} - l_{61+}^{(\tau)} = 1000 - 60 - 45 - 466 = 429$$

Graders' Comments: *There are different ways of doing this part; full credit was awarded for any correct method. Quite a few candidates gave the answer as 451, which is obtained by ignoring decrement (2).*

(c)

(i) If decrement 2 occurs at the start of the year, there are fewer lives exposed to force of decrement 1, so $q_{60}^{(1)}$ would be smaller.

(ii) If decrement 2 occurs at the start of the year, there are more lives exposed to force of decrement 2, so $q_{60}^{(2)}$ would be bigger.

(iii) Because all the decrement 3 exits happen at the end of the year, we have

$$q_{60}^{(3)} = \frac{d_{60}^{(3)}}{l_{60}^{(\tau)}} \qquad q_{60}'^{(3)} = \frac{d_{60}^{(3)}}{l_{61^-}^{(\tau)}}$$

Where $l_{61^-}^{(\tau)}$ is the expected number of in-force immediately before the decrement 3 exits at the end of the year. Also $l_{61^-}^{(\tau)} = l_{60}^{(\tau)} p_{60}'^{(1)} p_{60}'^{(2)}$ and since the independent rates are unchanged, $l_{61^-}^{(\tau)}$ is unchanged, which means that $d_{60}^{(3)}$ is unchanged, which means that $q_{60}^{(3)}$ is unchanged.

Question 3

(a) Death in Year 1:

$L_0 \mid Event = (1000 + G(1+i))v - G = 1000v = 943.4$

Probability $= 0.06$

Withdrawal in Year 1:

$L_0 \mid Event = -G$

Probability $= 0.04$

Death in Year 2:

$L_0 \mid Event = (1000 + G(1+i) + G(1+i)^2)v^2 - G(1+v) = 1000v^2 = 890.0$

Probability $= (0.90)(0.12) = 0.108$

Survival in force to end of year 2:

$L_0 \mid Event = -G(1+v) = -1.9434G$

Probability $= (0.9)(0.88) = 0.792$

In table form:

Event	Value of L_0, Given that the Event Occurred	Probability of Event
Death in year 1	943.4	0.06
Withdrawal in year 1	$-G$	0.04
Death in year 2	890.0	0.108
Neither death or withdrawal	$-1.9434G$	0.792

Graders' Comments: *Most candidates did well in this part. Amongst candidates who did not achieve full marks, the most common problem was determining the amount of the return of premiums benefit.*

(b) (i)

$$E[L_0] = (943.4)(0.06) + (-G)(0.04) + (890.0)(0.108) + (-1.9434G)(0.792)$$

$$= 152.7 - 1.579G$$

$$\Rightarrow a = 152.7 \quad \text{and} \quad b = 1.579$$

(ii)

$$E[L_0^2] = (943.4)^2(0.06) + (-G)^2(0.04) + (890.0)^2(0.108) + (-1.9434G)^2(0.792)$$

$$= 138{,}947 + 3.0312G^2$$

$$E[L_0]^2 = (152.7 - 1.579G)^2 = 23{,}317 - 482.2G + 2.4932G^2$$

$$V[L_0] = E[L_0^2] - E[L_0]^2 = 115{,}630 + 482.2G + 0.538G^2$$

$$\Rightarrow c = 0.538 \quad d = 482.2 \quad e = 115{,}630$$

Graders' Comments: *The table in part (a) was used by many candidates to answer part (b), as the examiners intended. Standard variance formulas for level benefit term insurance do not work in this case, and candidates who tried to use memorized formulas received no credit for this part.*

(c) For each policy:

$$E[L_0] = a - bG = -52.57$$

$$V[L_0] = cG^2 + dG + e = 187{,}408$$

So for the aggregate loss

$$E[L_{agg}] = (200)(-52.57) = -10{,}514$$

$$V[L_{agg}] = (200)(187{,}408) = 37{,}481{,}600 = (6122.2)^2$$

$$\Rightarrow \Pr[L_{agg} > 0] = 1 - \Phi\left(\frac{0-(-10{,}514)}{6122.2}\right) = 1 - \Phi(1.72)$$

$$= 1 - 0.9573 = 0.0427$$

Question 4

(a)

(i)

$${}_t p_x = \exp\left(-At - \frac{B}{\log c} c^x \left(c^t - 1\right)\right) \text{ from formula sheet.}$$

$$\Rightarrow {}_{0.5} p_{64} = e^{-0.00257678} = 0.9974265$$

$$\Rightarrow {}_{0.5} q_{64} = 0.0025735 \Rightarrow 100,000 \, {}_{0.5} q_{64} = 257.35$$

(ii)

$${}_4 V = \text{EPV(Benefit)} - \text{EPV(Premiums)}$$

$$= 100,000\left({}_{0.5} q_{64} \, v^{0.5} + {}_{0.5|0.5} q_{64} v\right) - 0.9(270)\left(1 + {}_{0.5} p_{64} \, v^{0.5}\right)$$

where $\quad {}_{0.5|0.5} q_{64} = q_{64} - {}_{0.5} q_{64} = 0.005288 - 0.0025735 = 0.0027145$

$$\Rightarrow {}_4 V = 100,000\left(0.0025115 + 0.0025852\right) - 243\left(1.9733886\right) = 30.14$$

Alternatively, by recursion

$$\left({}_{4.5} V + 0.9P\right)(1.05)^{0.5} = 100,000 \, {}_{0.5} q_{64.5} + {}_{0.5} p_{64.5} \; {}_5 V$$

$$\Rightarrow {}_{4.5} V = 272.15 v^{-0.5} - 0.9(270) = 22.5912$$

$$\left({}_4 V + 0.9P\right)(1.05)^{0.5} = 100,000 \, {}_{0.5} q_{64} + {}_{0.5} p_{64} \; {}_{4.5} V$$

$$\Rightarrow {}_{4.} V = 30.14$$

***Graders' Comments**: Most candidates knew to use either EPV or recursive formula to compute the reserve. Common errors made by candidates include:*

- *Not properly handling the semi-annual premiums;*
- *Approximating semi-annual annuity using annual annuity; and*
- *Assuming uniform distribution of deaths or constant force of mortality.*

(b) $\quad {}_1 p_{64}^{\overline{00}} = \exp\left\{-\int_0^1 \mu_{64+t}^{01} + \mu_{64+t}^{03} \, dt\right\} = e^{-0.06} \times p_{64}^{SULT} = 0.9367845$

(c)

$$ {}_4V^{(0)} = 100{,}000\left({}_{0.5}p_{64}^{03}\, v^{0.5} + \left(p_{64}^{03} - {}_{0.5}p_{64}^{03} \right) v \right) - 0.9(270)\left(1 + \left(1 - {}_{0.5}p_{64}^{03}\right) v^{0.5}\right) $$

$$ {}_{0.5}p_{64}^{03} = 1 - 0.00264 = 0.99736 $$

So $\;{}_4V^{(0)} = 100{,}000\left(0.002576 + 0.002697\right) - 243\left(1.97332\right) = 47.80$

***Graders' Comments**: Candidates generally earned some credit. Typical mistakes included:*

- *Did not consider all possible transitions when calculating the probability for the year-end benefit;*
- *Did not realize premium is paid in State 0, 1, and 2 or improperly calculated the EPV; and*
- *Did not consider state dependent reserves when using the recursive formula in a multiple state model.*

(d)

The expected time of each sojourn in "At risk", in years, is

$$ \int_0^\infty {}_t p_x^{\overline{11}}\, dt = \int_0^\infty e^{-t\left(\mu^{10}+\mu^{12}\right)} dt = \int_0^\infty e^{-100t} dt = \frac{1}{100} \text{ years} = 3.65 \text{ days} $$

***Graders' Comments**: Many candidates skipped this part.*

Question 5

(a) (i) Let $\mu^{13} = \mu_t^{13}$ represent the constant transition force from State 1 to State 3. Then we have

$$\bar{a}_x^{01} = \int_0^\infty {}_t p_0^{01}\, e^{-\delta t} dt$$

$$\bar{A}_0^{03} = \int_0^\infty {}_t p_0^{01}\, \mu_t^{13}\, e^{-\delta t} dt = \mu^{13} \int_0^\infty {}_t p_0^{01}\, e^{-\delta t} dt$$

$$\Rightarrow \frac{\bar{A}_0^{03}}{\bar{a}_x^{01}} = \mu^{13}$$

(ii)

$$\bar{a}_t^{\overline{11}} = \int_0^\infty {}_r p_t^{\overline{11}}\, e^{-\delta r} dr \quad \text{where} \quad {}_r p_t^{\overline{11}} = e^{\int_0^r \mu_{t+s}^{12} + \mu_{t+s}^{13} ds} = e^{-0.3r}$$

$$\Rightarrow \bar{a}_t^{\overline{11}} = \int_0^\infty e^{-0.34r} dr = \frac{1}{0.34} = 2.941$$

(iii) $\bar{a}_t^{\overline{11}} < \bar{a}_t^{11}$ since $\bar{a}_t^{11}$ also includes the (positive) EPV of payments made on returning to State 1 after visiting State 2.

Graders' Comments*: Most candidates completed Part i correctly. For all parts, some candidates had minor expression errors where their variables and/or integral limits were not consistent.*

Almost all candidates were able to correctly justify that annuity factor $\bar{a}_t^{\overline{11}}$ is smaller than $\bar{a}_t^{11}$.

(b)

(i) $22{,}000\, \bar{A}_0^{03} = 22{,}000\, (0.1)\, \bar{a}_0^{01}$ from (a) (i)

$= 22{,}000\ (0.1)\ (2.930) = 6446$

(ii) $12{,}000\, \bar{a}_0^{01} = 12{,}000\ (2.930) = 35{,}160$

(iii) The EPV is the sum of EPV of additional costs during first sojourn in State 1 (from State 0) up to 4 months, plus the EPV of additional costs during returning sojourns in State 1 from State 2 up to 4 months. That is:

$$EPV = 8000\left(\int_0^\infty {}_t p_0^{00}\, \mu_t^{01}\, \bar{a}^{\overline{11}}_{t:\overline{\frac{1}{3}}|}\, e^{-\delta t} dt + \int_0^\infty {}_t p_0^{02}\, \mu_t^{21}\, \bar{a}^{\overline{11}}_{t:\overline{\frac{1}{3}}|}\, e^{-\delta t} dt \right)$$

$$\bar{a}^{\overline{11}}_{t:\overline{\frac{1}{3}}|} = \int_0^{\frac{1}{3}} {}_r p_t^{\overline{11}}\, e^{-\delta r} dr = \frac{1-e^{-0.34/3}}{0.34} = 0.3151$$

(c) The expected time of the first transition to State 1 is unchanged.

i) **Decrease**
With a larger value of μ_t^{13}, the **expected time spent in State 1 will decrease.** With less time spent in State 1, the EPV of the cost in nursing care in State 1 will **decrease**.

ii) **Increase**
With a larger value of μ_t^{13}, the transition to State 3 **(surgery) will happen sooner, on average**. With surgery happening earlier, the EPV of the cost of surgery will **increase**.

***Graders' Comments**: For part (i), most candidates were able to answer and justify that higher μ_t^{13} will cause higher probability (and thus shorter time) from State 1 to State 3. For part (ii), while most candidates were able to correctly answer that a higher μ_t^{13} will increase EPV of surgery, they were not able to clearly express that the increased cost was due to shorter travel time to State 3 (absorbing state) and thus the EPV with less discounting is higher.*

Question 6

(a) Note that μ_{x+t}^{01} is the force of mortality under Makeham's Law with parameters A, B and c* , where $A = 0.004,\ B = 0.015,$ and $c^* = e^{0.005} = 1.0050125$

$${}_5p_{60}^{00} = e^{-5A - \frac{B}{\log c^*}(c^*)^{60}((c^*)^5 - 1)} = 0.8846922$$

Prob. of diagnosis $= 1 - {}_5p_{60}^{00} = 0.11531$

(b)

$$
\begin{aligned}
{}_tp_x^{12} &= \int_0^t {}_rp_x^{11}\, \mu_{x+r}^{12}\, {}_{t-r}p_{x+r}^{22}\, dr \\
&= \int_0^t e^{-0.2r}(0.2)e^{-0.4(t-r)}\, dr \\
&= \left(0.2\, e^{-0.4t}\right)\int_0^t e^{0.2r}\, dr = \left(0.2\, e^{-0.4t}\right)\left(\frac{e^{0.2t} - 1}{0.2}\right) \\
&= e^{-0.2t} - e^{-0.4t}
\end{aligned}
$$

(c) $${}_5p_x^{13} = 1 - {}_5p_x^{11} - {}_5p_x^{12} = 1 - e^{-5(0.2)} - \left(e^{-5(0.2)} - e^{-5(0.4)}\right) = 0.399577$$

(d)

(i) $$\bar{a}_{x:\overline{5}|}^{11} = \int_0^5 {}_tp_x^{11}\, e^{-\delta t} dt = \int_0^5 e^{-0.2t} e^{-0.05t} dt$$

$$= \frac{1 - e^{5(0.25)}}{0.25} = 2.85398$$

(ii) $$\bar{a}_{x:\overline{5}|}^{12} = \int_0^5 {}_tp_x^{12}\, e^{-\delta t} dt = \int_0^5 \left(e^{-0.2t} - e^{-0.4t}\right)e^{-0.05t} dt$$

$$= 2.85398 - \frac{1 - e^{5(0.45)}}{0.45} = 0.86598$$

(iii) $$\bar{a}_{x:\overline{5}|}^{11} + \bar{a}_{x:\overline{5}|}^{12} + \bar{a}_{x:\overline{5}|}^{13} = \int_0^5 \left({}_tp_x^{11} + {}_tp_x^{12} + {}_tp_x^{13}\right) e^{-\delta t}\, dt$$

$$= \int_0^5 1\, e^{-\delta t}\, dt \text{ (law of complete probability)}$$

$$= \bar{a}_{\overline{5}|}$$

Graders' Comments: *Candidates who attempted this part did well on this question in general, with most receiving full marks. Part iii) required a rigorous demonstration.*

(e)

Under Option A the QL index is $0.5\,\bar{a}^{11}_{x:\overline{5}|} + 0.2\,\bar{a}^{12}_{x:\overline{5}|} + 1.0\,\bar{a}^{13}_{x:\overline{5}|}$, where the annuities are now calculated with $\mu^{23}_x = 1.0$.

$\bar{a}^{11}_{x:\overline{5}|} = 2.85398$ from above.

$${}_tp^{12}_x \text{ is now } = \int_0^t e^{-0.2r}(0.2)e^{-(t-r)}\,dr$$
$$= \left(0.2\,e^{-t}\right)\left(\frac{e^{0.8t}-1}{0.8}\right)$$
$$= 0.25\left(e^{-0.2t} - e^{-t}\right)$$

So $\bar{a}^{12}_{x:\overline{5}|} = 0.25\int_0^5 \left(e^{-0.2t} - e^{-t}\right)e^{-0.05t}dt$
$$= 0.25\left(2.85398 - \frac{1-e^{-(1.05)5}}{1.05}\right)$$
$$= 0.47665$$

Also $\bar{a}^{13}_{x:\overline{5}|} = \bar{a}_{\overline{5}|} - \bar{a}^{11}_{x:\overline{5}|} - \bar{a}^{12}_{x:\overline{5}|}$
$$= 4.42398 - 2.85398 - 0.47665$$
$$= 1.09335$$

So $\text{QL}^A = 0.5(2.85398) + 0.2(0.47665) + 1.0(1.09335)$
$$= 2.61567$$

Under Option B the QL index is $0.55\,\bar{a}^{11}_{x:\overline{5}|} + 0.3\bar{a}^{12}_{x:\overline{5}|} + 1.0\,\bar{a}^{13}_{x:\overline{5}|} = 2.534$, where the annuities are calculated as in part (d), with

$\bar{a}^{12}_{x:\overline{5}|} = 0.86598$ and $\bar{a}^{13}_{x:\overline{5}|} = 4.42398 - 2.85398 - 0.86598 = 0.70402$

$\Rightarrow QL^B = 2.5335$

Based on the improved QL, the health authority should select Option A.

Graders' Comments: *This problem was not attempted by a majority of the candidates; those that did generally received most of the available credit.*

Question 7

(a)

$$\frac{d}{dt}\ {}_tp_0^{00} = {}_tp_0^{01}\mu_t^{10} - {}_tp_0^{00}\mu_t^{01}$$

$$\frac{d}{dt}\ {}_tp_0^{01} = {}_tp_0^{00}\mu_t^{01} - {}_tp_0^{01}\left(\mu_t^{10} + \mu_t^{12}\right)$$

$$\frac{d}{dt}\ {}_tp_0^{02} = {}_tp_0^{01}\mu_t^{12}$$

Boundary Conditions:

$${}_0p_0^{00} = 1 \quad {}_0p_0^{01} = 0 \quad {}_0p_0^{02} = 0$$

Graders' Comments: *Most candidates were able to give the differential equations, but quite a few did not provide boundary conditions.*
For full credit, candidates were expected to use the specific model given in the question, which meant that subscripts and superscripts needed to correspond with the notation of the question for full credit.

(b) (i)

The Euler equation for ${}_{t+h}p_0^{01}$ is:

$${}_{t+h}p_0^{01} = {}_tp_0^{01} + h\left({}_tp_0^{00}\mu_t^{01} - {}_tp_0^{01}\left(\mu_t^{10} + \mu_t^{12}\right)\right)$$

so, using $t = 0, h = 0.5$ gives:

$$\Rightarrow\ {}_{0.5}p_0^{01} = {}_0p_0^{01} + h\left({}_0p_0^{00}\mu_0^{01} - {}_0p_0^{01}\left(\mu_0^{10} + \mu_0^{12}\right)\right)$$

$$= 0 + 0.5\left((1)(0.5) + (0)\left(0.2 + 2^0\right)\right) = 0.25$$

(ii)

The Euler equation for ${}_{t+h}p_0^{02}$ is

$${}_{t+h}p_0^{02} = {}_tp_0^{02} + h\left({}_tp_0^{01}\mu_t^{12}\right)$$

so, using $t = 0.0$, $h = 0.5$ and then $t = 0.5, h = 0.5$ gives

$${}_{0.5}p_0^{02} = {}_0p_0^{02} + h\left({}_0p_0^{01}\mu_0^{12}\right) = 0 + 0.5\left((0)(2^0)\right) = 0$$

and $${}_1p_0^{02} = {}_{0.5}p_0^{02} + h\left({}_{0.5}p_0^{01}\mu_{0.5}^{12}\right) = 0 + 0.5\left((0.25)(2^{0.5})\right) = 0.177.$$

Graders' Comments*:* *Many candidates' solutions lacked clarity in this part, which made it difficult to award partial credit for incomplete answers.*

(c)

(i) $APV = 1000(v \times {}_{0.5}p_0^{02} + v^2({}_1p_0^{02} - {}_{0.5}p_0^{02})$ at 4%

$$=1000\left[(1.04)^{-1}(0)+(1.04)^{-2}(0.177-0)\right]=163.64$$

(ii) $P(1+v \times {}_{0.5}p_0^{00}) = APV = 163.64$

$$\Rightarrow P = \frac{163.64}{1+(1.04)^{-1}(0.75)} = 95.08$$

Graders' Comments: *This part was done well by most candidates who attempted it. Note that all the probability values required were given in the question, so it was not necessary to answer (b) to answer (c) correctly. Full credit was given for answers using the rounded probability values given in the question.*

(d) Use smaller *h* for greater accuracy.

Question 8

(a)

$$\bar{a}^{0j}_{x:\overline{10|}} = \int_0^{10} {}_t p_x^{0j} \cdot e^{-\delta t} \cdot dt$$

$$\Rightarrow \bar{a}^{00}_{x:\overline{10|}} + \bar{a}^{01}_{x:\overline{10|}} + \bar{a}^{02}_{x:\overline{10|}} = \int_0^{10} ({}_t p_x^{00} + {}_t\, p_x^{01} + {}_t\, p_x^{02}) \cdot e^{-\delta t} \cdot dt$$

but ${}_t p_x^{00} + {}_t\, p_x^{01} + {}_t\, p_x^{02} = 1$

$$\Rightarrow \bar{a}^{00}_{x:\overline{10|}} + \bar{a}^{01}_{x:\overline{10|}} + \bar{a}^{02}_{x:\overline{10|}} = \int_0^{10} e^{-\delta t} \cdot dt = \bar{a}_{\overline{10|}}$$

Graders' Comments: *Many candidates did well on this part. Some candidates gave a verbal explanation, rather than mathematical proof. Generally, this earned partial credit, but full credit was awarded if the verbal proof was sufficiently detailed.*

(b)

EPV Premiums: $P\bar{a}^{00}_{x:\overline{10|}} = 4.49P$

EPV Disability Annuity: $1000\bar{a}^{01}_{x:\overline{10|}} = 1000\left(\bar{a}_{\overline{10|}} - (\bar{a}^{00}_{x:\overline{10|}} + \bar{a}^{02}_{x:\overline{10|}})\right)$ where $\bar{a}_{\overline{10|}} = \dfrac{1-v^{10}}{\delta} = 6.32121$

$\Rightarrow EPV$ Disability Annuity $= 471.21$

EPV Death Benefit: $10000\bar{A}^{02}_{x:\overline{10|}} = 3871.0$

So the Premium is $P = \dfrac{3871 + 471.21}{4.49} = 967.1$

(c) The Thiele equation at time t is

$$\frac{d}{dt}\,{}_tV^{(0)} = \delta \cdot {}_tV^{(0)} + P - \mu_{x+t}^{01}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu_{x+t}^{02}\left(10{,}000 - {}_tV^{(0)}\right)$$

so at $t = 3$ we have

$$\left.\frac{d}{dt}\,{}_tV^{(0)}\right|_{t=3} = (0.1)(1304.54) + 967.1$$
$$-0.04\left(7530.09 - 1304.54\right)$$
$$-3(0.2)\left(10{,}000 - 1304.54\right)$$
$$= 326.80$$

Graders' Comments: *This part proved more challenging to many candidates. The responses indicated that many candidates are memorizing Thiele's formula rather than understanding the intuition behind it.*

(d) Let P^* denote the new premium. The EPV of the return of premium benefit is

$\text{EPV}=10P^* e^{-10\delta}\,{}_{10}p_x^{\overline{00}}$, where

$${}_{10}p_x^{\overline{00}} = \exp\left(-\int_0^{10} \mu_{x+t}^{01} + \mu_{x+t}^{02}\,dt\right) = \exp\left(-\int_0^{10} 0.04 + 0.02t\;\,dt\right) = e^{-1.4} = 0.24660$$

$$\Rightarrow \text{EPV} = 10P^* e^{-10\times 0.1}(0.24660) = 0.90718P^*$$

$$\Rightarrow P^* = \frac{3871 + 471.2}{4.490 - 0.90718} = 1212.0$$

$$\Rightarrow \left(P^* - P\right) = 244.9$$

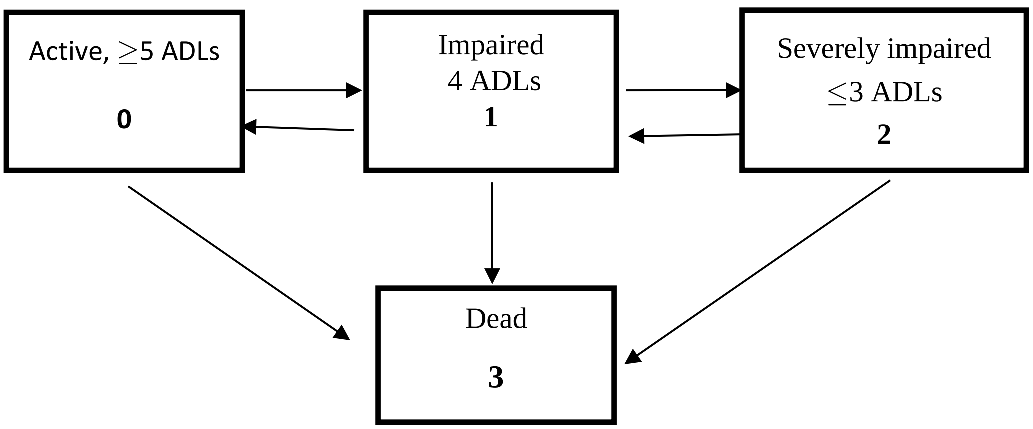
Graders' Comments: *A common error was to omit the discount factor, $e^{-10\delta}$, possibly on the grounds that the premiums are returned without interest, but the discount factor reflects the interest earned by the insurer, not the interest that might be returned to the policyholder. If the premiums were returned with interest, the benefit would be in the form $P^*\bar{s}_{\overline{10|}}$ instead of $10P^*$.*

Question 9

(a) The six common ADLs are

- Bathing
- Dressing
- Eating
- Toileting
- Continence
- Transferring

(b) One model is given below. Others may also be appropriate.



(c)

(i) Over the course of the policy, the insured was active for 12 + 12 + 12 + 6 + 4 + 12 +12 + 12 + 12 + 6 = 100, months, thereby paying a total of 100 × 150 = 15,000 in premium. For the first disability, (6 - 3) × 1000 = 3000 in benefits were paid; for the second disability, (8-3) × 2000 = 10,000 in benefits were paid. Since the sum of the premiums paid exceeds the disability benefits paid out under the policy, under the “return of premium" approach, the remainder of 15,000-13,000 = 2000 is added to the death benefit, for a total death benefit of 102,000.

(ii) As before, over the course of the policy, 13,000 in benefits were paid. This amount is deducted from the sum insured of 100,000, leaving a death benefit payment of 87,000.

(d)

(i) With a 12 month off period, the second disability, which starts 9 months after the end of the first disability period, is not subject to the 3 month waiting period. Hence, benefit payments will commence immediately for this disability, increasing the long term care benefits paid by $3 \times 2000 = 6000$. However, since the total long term care benefits paid (19,000) would now exceed the total premium paid (15,000), there would no longer be a return of premium added to the death benefit, lowering it by 2000. Hence, the total benefits paid would increase by 4000.

(ii) Again, in this scenario, the long term care benefits paid would increase by 6000. However, the death benefit paid would be 6,000 less than before. Hence, there is no net change in total benefits paid by the policy.

Question 10

(a)

$$ {}_{t+h}p_x^{12} = {}_t p_x^{11}\, {}_h p_{x+t}^{12} + {}_t p_x^{12}\, {}_h p_{x+t}^{22} \quad \text{(Markov property)} $$

$$ = {}_t p_x^{11}\, {}_h p_{x+t}^{12} + {}_t p_x^{12}\left(1 - {}_h p_{x+t}^{23} - {}_h p_{x+t}^{24}\right) \quad \text{(complete probability)} $$

$$ = {}_t p_x^{12} + {}_t p_x^{11}\, {}_h p_{x+t}^{12} - {}_t p_x^{12}\, {}_h p_{x+t}^{23} - {}_t p_x^{12}\, {}_h p_{x+t}^{24} $$

$$ \Rightarrow \frac{{}_{t+h}p_x^{12} - {}_t p_x^{12}}{h} = {}_t p_x^{11}\left(\frac{{}_h p_{x+t}^{12}}{h}\right) - {}_t p_x^{12}\left(\frac{{}_h p_{x+t}^{23}}{h}\right) - {}_t p_x^{12}\left(\frac{{}_h p_{x+t}^{24}}{h}\right) $$

and, where the transition intensity exists, $\mu_{x+t}^{ij} = \lim_{h\to 0}\frac{{}_h p_{x+t}^{ij}}{h}$, and taking limits gives

$$ \frac{d}{dt}\,{}_t p_x^{12} = {}_t p_x^{11}\mu_{x+t}^{12} - {}_t p_x^{12}\mu_{x+t}^{23} - {}_t p_x^{12}\mu_{x+t}^{24} $$

(b)

$$ {}_t p_x^{12} = \int_0^t {}_r p_x^{11}\, \mu_{x+r}^{12}\, {}_{t-r}p_{x+r}^{22}\, dr = \int_0^t e^{-0.24r} \times 0.10 \times e^{-0.34(t-r)} dr $$

$$ = 0.10(e^{-0.34t})\int_0^t e^{0.1r} dr = 0.10(e^{-0.34t})\frac{e^{0.1t}-1}{0.1} = e^{-0.24t} - e^{-0.34t} $$

(c)

The EPV is $3000\bar{a}_{90}^{12}$ where

$$ \bar{a}_{90}^{12} = \int_0^\infty {}_t p_{90}^{12}\, e^{-\delta t} dt = \int_0^\infty \left(e^{-0.24t} - e^{-0.34t}\right) e^{-0.05t} dt $$

$$ = \int_0^\infty \left(e^{-0.29t} - e^{-0.39t}\right) dt = \frac{1}{0.29} - \frac{1}{0.39} $$

$$ = 0.88417 $$

So the EPV is $3000(0.88417) = 2652.5$.

(d)

$$ \bar{a}_{90:\overline{0.5}|}^{\overline{22}} = \int_0^{0.5} {}_t p_{90}^{\overline{22}}\, e^{-\delta t} dt = \int_0^{0.5} e^{-0.34t}\, e^{-0.05t} dt = \frac{1 - e^{-0.5\times 0.39}}{0.39} = 0.4543 $$

(e) The EPV of a benefit of 1 per year payable continuously after the waiting period is

$$\int_0^{\infty} {}_r p_{90}^{11}\, \mu_{90+r}^{12} \left(\bar{a}_{90+r}^{\overline{22}} - \bar{a}_{90+r:\overline{0.5|}}^{\overline{22}} \right) e^{-\delta r} dr \quad \text{where} \quad \bar{a}_{90+r}^{\overline{22}} = \frac{1}{0.39} = 2.5641$$

So the EPV is

$$\int_0^{\infty} e^{-0.24r} \times 0.10 \times (2.5641 - 0.4542) e^{-0.05r} dr = 0.21099 \int_0^{\infty} e^{-0.29r} dr = \frac{0.21099}{0.29}$$

$$=0.7276$$

So for a benefit of 3000 per year the EPV is 3000×0.7276=2182.7.

(f) Reason 1: Short term payments involve relatively high expenses. Once the illness has extended beyond the waiting period, it is likely to be more significant and less costly relative to the benefits

Reason 2: In some cases the policyholders will have other sources of income for short term sickness, eg sick pay from employment.

Reason 3: Offer policyholders a choice, for the same premium they will receive higher benefits for long term sickness in return for giving up benefit for shorter bouts.

Question 11

(a)(i) EPV of future costs is

$$12\times 3000\times \ddot{a}_{65}^{(12)00}+12\times 7500\times \ddot{a}_{65}^{(12)01}+12\times 15000\times \ddot{a}_{65}^{(12)02}=600{,}321.6$$

So F is 25% of the EPV = 150,080

(a)(ii) The monthly fee is M where

$$12M\,(\ddot{a}_{65}^{(12)00}+\ddot{a}_{65}^{(12)01}+\ddot{a}_{65}^{(12)02})=0.75(600321.6)\Rightarrow M=2854.95$$

(b)(i) The reserve is EPV future outgo – EPV future fee income, so

$$\begin{aligned}{}_5V^{(0)}&=12(3000)\ddot{a}_{70}^{(12)00}+12(7500)\ddot{a}_{70}^{(12)01}+12(15000)\ddot{a}_{70}^{(12)02}\\&\qquad-12(2854.95)\left(\ddot{a}_{70}^{(12)00}+\ddot{a}_{70}^{(12)01}+\ddot{a}_{70}^{(12)02}\right)\\&=183{,}563\end{aligned}$$

(b)(ii) Now we have

$$\begin{aligned}{}_5V^{(1)}&=12\times 7500\times \ddot{a}_{70}^{(12)11}+12\times 15000\times \ddot{a}_{70}^{(12)12}-12\times 2854.95\times\left(\ddot{a}_{70}^{(12)11}+\ddot{a}_{70}^{(12)12}\right)\\&=712{,}340\end{aligned}$$

(c) By recursion, we have

$$\left({}_{4\frac{11}{12}}V^{(0)}+M-3000\right)(1.05)^{\frac{1}{12}}={}_{\frac{1}{12}}p_{69\frac{11}{12}}^{00}\;{}_5V^{(0)}+{}_{\frac{1}{12}}p_{69\frac{11}{12}}^{01}\;{}_5V^{(1)}+{}_{\frac{1}{12}}p_{69\frac{11}{12}}^{02}\;{}_5V^{(2)}$$

$${}_5V^{(2)}=12\times(15000-2854.95)\times \ddot{a}_{70}^{(12)22}=1{,}340{,}245$$

$$\begin{aligned}\Rightarrow\ {}_{4\frac{11}{12}}V^{(0)}&=\left(0.94937(183{,}563)+0.00906(712{,}340)+0.00003(1{,}340{,}245)\right)v^{\frac{1}{12}}+145.05\\&=180{,}175\end{aligned}$$

(d)(i) The equation of value for F is now

$$F=0.25\left(600{,}321.6+0.5FA_{65}^{(12)03}\right)=150{,}080+0.04501F$$

$$\Rightarrow F=157{,}153$$

(d)(ii) The equation of value for M is now

$$12M\,(\ddot{a}_{65}^{(12)00}+\ddot{a}_{65}^{(12)01}+\ddot{a}_{65}^{(12)02})=0.75(600321.6+0.5FA_{65}^{(12)03})$$

$$\Rightarrow M=2989.52$$

Question 12

(a) 1) To better replicate the loss to the injured party (*IP*):

- lost wages, medical and other expenses due to injury, offset inflation (if increasing annuity).

2) To relieve the *IP* from the investment/interest risk, or from the burden of managing funds.

3) To reduce the dissipation risk:

- risk of running out of funds from squandering/over spending.

(b) 1) Tax :

The structured settlement annuity might not be taxed, or taxed at a lower rate than salary. Less than 100% is needed to replace the net pre-injury earnings.

2) Incentive to return to work:

The *IP* may choose to return to work if that would increase his/her (net) income.

3) *IP* at fault:

The payments may be reduced if the *IP* is (partially) at fault.

Graders' Comments: *Performance on parts (a) and (b) was mixed. Some candidates did not answer those two parts, others provided only one relevant reason for each part and/or repeated the same reasons for both parts. A number of candidates discussed pricing or underwriting issues, in one or both parts, for which no credit was awarded.*

(c) The benefits are 90,000 when in States 0 or 2 plus 20,000 when in State 0 for up to 2 years. So,

$$EPV = 90,000\left(\bar{a}_x^{00} + \bar{a}_x^{02}\right) + 20,000\,\bar{a}_{x:\overline{2|}}.$$

Since returning to State 0 after leaving it is impossible, we have

$$\bar{a}_{x:\overline{2|}}^{00} = \bar{a}_x^{00} - v^2\,{}_2p_x^{00}\,\bar{a}_{x+2}^{00} = 0.543444$$

$$\Rightarrow EPV = 90,000\left(0.559 + 7.161\right) + 20,000(0.543444) = 705,669$$

(d)

(i) $\quad {}_2V^{(0)} = 90,000\left(\bar{a}_{x+2}^{00} + \bar{a}_{x+2}^{02}\right) = 698,670$

(ii) $\quad {}_2V^{(2)} = 90,000\,\bar{a}_{x+2}^{22} = 956,070$

(iii) Let $\mathrm{E}_0\left[{}_2V\right]$ denote the EPV at $t=0$ of the policy value at $t=2$. Note that there is no policy value in State 1 or State 3. Then
$\mathrm{E}_0\left[{}_2V\right] = v^2\,{}_2p_x^{00}\,{}_2V^{(0)} + v^2\,{}_2p_x^{02}\,{}_2V^{(2)} = 564,054$

(iv) The EPV at $t=0$ of payments in the first two years plus the EPV at $t=0$ of the policy value at $t=2$ is equal to the EPV at $t=0$ of all future payments. That is
$\text{EPV(First two years payments)} = EPV - E_0\left[{}_2V\right] = 141,615$

Graders' Comments: *Most candidates did well on this part. Some candidates calculated the EPV in (iv) directly, which was an acceptable alternative solution.*

(e)

(i) The reserve at time 2 given State 2, ${}_2V^{(2)}$ will stay the same.
This reserve is conditional on *IP* being in State 2 and transitions out of State 2 are not affected.

(ii) Increasing μ_{x+t}^{01} will lead to a decrease in ${}_tp_x^{00}$ as lives are more likely to move from State 0 to State 1 in the interval from time 0 to time *t*. It will also lead to a decrease in ${}_tp_x^{02}$, as it means that lives are more likely to recover, and so less likely to become permanently disabled. That means that ${}_2V^{(0)}$ will be smaller, and hence $\mathrm{E}_0\left[{}_2V\right] = v^2\,{}_2p_x^{00}\,{}_2V^{(0)} + v^2\,{}_2p_x^{02}\,{}_2V^{(2)}$ will also be smaller.

Graders' Comments: *Most candidates determined that the reserve would stay the same in part (i). Part (ii) proved to be more challenging. Candidates were expected to address each of the terms in* $\mathrm{E}_0\left[{}_2V\right]$.

Question 13

(a) EPV(Death Benefit) $= 50,000 A_{55}^{(12)02} = 24,000$

EPV(Premiums $-$ Commissions) $= 0.95 G \ddot{a}_{55}^{00} = 8.3904G$

$\Rightarrow 24,000 - 8.3904G = -0.1G$

$\Rightarrow G = 2894.9$

(b) ${}_{10}V^{(0)} = 50,000 A_{65}^{(12)02} - 0.95 G \ddot{a}_{65}^{00} = 16,805$

(c) (i) $\left({}_{10}V^{(0)} + 0.95G\right)(1.05)^{\frac{1}{12}} = {}_{\frac{1}{12}}p_{65}^{00}\ {}_{10\frac{1}{12}}V^{(0)} + {}_{\frac{1}{12}}p_{65}^{01}\ {}_{10\frac{1}{12}}V^{(1)} + {}_{\frac{1}{12}}p_{65}^{02}(50,000)$

${}_{\frac{1}{12}}p_{65}^{00} = 1 - {}_{\frac{1}{12}}p_{65}^{01} - {}_{\frac{1}{12}}p_{65}^{02} = 0.99246$

$\Rightarrow {}_{10\frac{1}{12}}V^{(0)} = 19,478$

(ii) $\left({}_{10\frac{1}{12}}V^{(0)}\right)(1.05)^{\frac{1}{12}} = {}_{\frac{1}{12}}p_{65\frac{1}{12}}^{00}\ {}_{10\frac{1}{12}}V^{(0)} + {}_{\frac{1}{12}}p_{65\frac{1}{12}}^{01}\ {}_{10\frac{1}{12}}V^{(1)} + {}_{\frac{1}{12}}p_{65\frac{1}{12}}^{02}(50,000)$

${}_{\frac{1}{12}}p_{65\frac{1}{12}}^{00} = 1 - {}_{\frac{1}{12}}p_{65\frac{1}{12}}^{01} - {}_{\frac{1}{12}}p_{65\frac{1}{12}}^{02} = 0.99238$

$\Rightarrow {}_{10\frac{2}{12}}V^{(0)} = 19,398$

Graders' Comments: *The majority of candidates got part (i) correct. Fewer candidates got part (ii) correct with the most common error being to include the premium.*

(d) (i) ${}_{10\frac{2}{12}}V^{(0)}$ is the EPV of future benefits – the EPV of future premiums net of commissions for a life in State 0 at $t = 10\frac{2}{12}$. The EPV of future benefits is unchanged, the EPV of future premiums net of commissions is smaller, so ${}_{10\frac{2}{12}}V^{(0)}$ will increase.

(ii) ${}_{10\frac{2}{12}}V^{(1)}$ is the EPV of future benefits – the EPV of future premiums net of commissions for a life in State 1 at $t = 10\frac{2}{12}$. The change in commissions does not affect this calculation, as no premiums are paid once the life transitions into State 1, as there are no transitions back to State 0.

Graders' Comments: *Candidates did not do as well on this part. Many candidates assumed that the reserve in part (i) would decrease. It is true that the accumulated past premiums minus commissions would be smaller, but prospectively, the EPV of future outgo minus income would be bigger. In (ii) most candidates correctly stated that the reserve would not change but did not offer a sufficient explanation.*

Question 14

(a) $$p_{40}^{(\tau)} = \exp\left\{-\int_0^1 \mu_{40+t}^{(\tau)}dt\right\} = \exp\left\{-4\int_0^1 \mu_{40+t}^{(1)}dt\right\} = \left(\exp\left\{-\int_0^1 \mu_{40+t}^{(1)}dt\right\}\right)^4$$

$$\exp\left\{-\int_0^1 \mu_{40+t}^{(1)}dt\right\} = \exp\left\{-A - \frac{B}{\log c}c^{40}(c-1)\right\} \ = p_{40}'^{(1)} = 0.9988415$$

(using Makeham's survival probability formula)

$$\Rightarrow p_{40}^{(\tau)} = \left(p_{40}'^{(1)}\right)^4 = 0.995374$$

Graders' Comments: *Candidates did well on this part. The most common error was treating decrements as independent instead of dependent. While this resulted in the same answer to four decimal places, points were deducted for an incorrect calculation. Another common error was assuming that the force of decrement was constant.*

(b)

$$\text{EPV} = 100,000\sum_{k=0}^{2} v^{k+1} \cdot {}_k\, p_x^{(\tau)} \cdot q_{x+k}^{(\tau)} = 100,000\sum_{k=0}^{2} v^{k+1}({}_k\, p_x^{(\tau)} - {}_{k+1}\, p_x^{(\tau)})$$

$$\text{EPV} = 100,000\left[\begin{array}{l}(1-0.99537)v + (0.99537-0.99027)v^2 \\ \qquad\qquad\qquad\qquad + (0.99027-0.98462)v^3\end{array}\right]$$

$$= 100,000(0.013916) = 1391.60$$

(i)

$$\text{EPV(premiums)} = P(1+0.99537v+0.99027v^2) = P(2.84618)$$

$$P = \frac{1391.60}{2.846179} = 488.93$$

(c)

(i) $${}_tq_{40}^{(1)} = \int_0^t {}_sp_{40}^{(\tau)}\,\mu_{40+s}^{(1)}\,ds$$

(ii) $${}_tq_{40}^{(\tau)} = \int_0^t {}_sp_{40}^{(\tau)}\,\mu_{40+s}^{(\tau)}\,ds = 4\int_0^t {}_sp_{40}^{(\tau)}\,\mu_{40+s}^{(1)}\,ds$$

$$= 4\,{}_tq_{40}^{(1)} \Rightarrow {}_tq_{40}^{(1)} = \frac{1}{4}\,{}_tq_{40}^{(\tau)}$$

(iii) $\text{EPV(Disease 1 rider)} = 50,000\sum_{k=0}^{k=2} v^{k+1}\,{}_{k|}q_{40}^{(1)}$

$$= \frac{50,000}{4}\sum_{k=0}^{k=2} v^{k+1}\,{}_{k|}q_{40}^{(\tau)}$$

$$= \frac{50,000}{4}\sum_{k=0}^{2} v^{k+1}({}_k p_x^{(\tau)} - {}_{k+1} p_x^{(\tau)})$$

$$= \frac{50,000 \times 0.013916}{4} = 173.95 \qquad \text{from (b)}$$

$$\text{Rider Premium } = P^r = \frac{173.95}{2.8462} = 61.12 \qquad \text{(denominator from (b))}$$

Graders' Comments *Part (i) was done well. In part (ii), candidates who assumed constant forces of decrement were given partial credit. Many candidates skipped (iii), but those who attempted it did well.*

(d) $\left({}_0V + P + P^r\right)(1.05) = 100,000q_{40}^{(\tau)} + 50,000q_{40}^{(1)} + p_{40}^{(\tau)}\,{}_1V$

$\Rightarrow {}_1V = 57.39$

Graders' Comments: *Most candidates who attempted this part calculated ${}_1V$ using the EPV of future outgo minus income. This is a longer method and tricky to get right, but is an acceptable alternative.*

Question 15

(a) EPV(Death Benefit): $20,000\bar{A}^{02}_{x:\overline{10}|} = 3020$

EPV(Sickness Benefit): $1000\,\bar{a}^{01}_{x:\overline{10}|} = 684$

EPV(expenses): $60\left(\bar{a}^{00}_{x:\overline{10}|} + \bar{a}^{01}_{x:\overline{10}|}\right) = 391.68$

EPV(Premiums – Commission): $0.95P\bar{a}^{00}_{x:\overline{10}|} = 5.5518P$

So $P = \dfrac{4095.68}{5.5518} = 737.72$

(b)

Pr(death within 10 years without becoming sick) is

$$\int_0^{10} {}_t p_x^{\overline{00}}\, \mu^{02}_{x+t}\, dt = \int_0^{10} 0.02e^{-(0.03+0.02)t}\, dt = 0.02\,\frac{1-e^{-0.5}}{0.05} = 0.15739$$

(c) The *elimination period* or *waiting period* is the time between the beginning of a period of disability/sickness and the start of the disability income payments.

(d) $$\bar{a}^{\overline{11}}_{x:\overline{1}|} = \int_0^1 {}_t p_x^{\overline{11}}\, e^{-\delta t}\, dt = \int_0^1 e^{-(0.01+0.05)t}\, e^{-0.07t}\, dt = \int_0^1 e^{-0.13t}\, dt$$

$$= \frac{1-e^{-0.13}}{0.13} = 0.93773$$

(e) The only change is the duration of the sickness benefit.

$$\text{EPV(Sickness benefit)} = 1000\int_0^{10} {}_t p_x^{00}\, \mu^{01}_{x+t}\left(\bar{a}^{\overline{11}}_{x+t} - \bar{a}^{\overline{11}}_{x+t:\overline{1}|}\right) e^{-\delta t}\, dt$$

$$\text{where } \bar{a}^{\overline{11}}_{x+t} = \int_0^{\infty} {}_s p_{x+t}^{\overline{11}}\, e^{-\delta s}\, ds = \int_0^{\infty} e^{-0.13s}\, ds = \frac{1}{0.13} = 7.6923$$

$\Rightarrow$ EPV(Sickness benefit)

$$= 1000(7.6923 - 0.93773)1000\int_0^{10} {}_t p_x^{00}\, \mu^{01}_{x+t}\left(\bar{a}^{\overline{11}}_{x+t} - \bar{a}^{\overline{11}}_{x+t:\overline{1}|}\right) e^{-\delta t}\, dt$$

$$= 6754.58\int_0^{10} {}_t p_x^{00}\, \mu^{01}_{x+t}\, e^{-\delta t}\, dt = 6754.58\bar{A}^{01}_{x:\overline{10}|} = 1182.05$$

$$\Rightarrow P = \frac{3020 + 1182.05 + 391.68}{5.5518} = 827.43$$

Graders' Comments: *This part was challenging, and few candidates achieved full marks, but many candidates achieved partial credit for answers that indicated some understanding.*

Question 16

(a) (i) $\bar{A}^{02}_{60:\overline{10|}} = \bar{A}^{02}_{60} - {}_{10}p^{00}_{60}\, v^{10}\, \bar{A}^{02}_{70} - {}_{10}p^{01}_{60}\, v^{10}\, \bar{A}^{12}_{70} =$

$$0.39077 - (0.75055)(1.05)^{-10}(0.54335) - (0.13135)(1.05)^{-10}(0.62237)$$

$$= 0.09022$$

(ii) ${}_{10}V^{(0)} = 500,000\ \bar{A}^{02}_{60:\overline{10|}} + 100,000\bar{A}^{01}_{60:\overline{10|}} - 4850\,\bar{a}^{00}_{60:\overline{10|}} = 21,850.4$

$$\bar{a}^{00}_{60:\overline{10|}} = \bar{a}^{00}_{60} - {}_{10}p^{00}_{60}\, v^{10}\, \bar{a}^{00}_{70} = 7.1461$$

$$\bar{A}^{01}_{60:\overline{10|}} = \bar{A}^{01}_{60} - {}_{10}p^{00}_{60}\, v^{10}\bar{A}^{01}_{70} = 0.11397$$

(iii) ${}_{10}V^{(1)} = 500,000\bar{A}^{12}_{60:\overline{10|}} = 95,700$

where $\bar{A}^{12}_{60:\overline{10|}} = \bar{A}^{12}_{60} -{}_{10}\ p^{11}_{60} v^{10} A^{12}_{70}$

$$= 0.47904 - 90.75283)(1.05)^{-10}(0.62237)$$

$$= 0.19140$$

(b)

(i) $\dfrac{d}{dt}\,{}_tV^{(0)} = \delta\,{}_tV^{(0)} + P - \mu^{01}_{x+t}\left(100,000 + {}_tV^{(1)} - {}_tV^{(0)}\right) - \mu^{02}_{x+t}\left(500,000 - {}_tV^{(0)}\right)$

At $t = 10$:

$$\frac{d}{dt}\,{}_tV^{(0)} = \ln(1.05)(21,850) + 4850$$
$$- (0.00818)(100,000 + 95,700 - 21,850)$$
$$- (0.00724)(500,000 - 21,850)$$
$$= 1032.2$$

(ii) $\dfrac{d}{dt}\,{}_tV^{(1)} = \delta\,{}_tV^{(1)} - \mu^{12}_{x+t}\left(500,000 - {}_tV^{(1)}\right)$

At $t = 10$:

$$\frac{d}{dt}\,{}_tV^{(1)} = \ln(1.05)(95,700) - (0.01811)(500,000 - 97,500)$$
$$= -2652.8$$

(iii) In State 0, the cash flows in are interest and premiums, and the cash flows out are claims and reserves on transition to State 1, and claims on transition to State 2.

In the early years of the term insurance the State 0 reserve will gradually increase, as premiums exceed expected outgo. In later years the reserve will decrease, as the greater number of claims outweighs the premium and investment income. At time 10, the reserve is still in the increasing phase.

If a policy is in state 1 it is effectively a single premium term insurance. There is a little interest income, but no premium income. The outgo on claims, on average, depletes the reserve, which will decline to 0 just before the end of the contract.

Graders' Comments: *Candidates found part (b) challenging. Many candidates skipped part (iii).*

(c)

$$100{,}000\int_0^{20} {}_t p_x^{00}\ \mu_{x+t}^{01}\ {}_{0.5}p_{x+t}^{11}\ v^{t+0.5}\ dt$$

Graders' Comments: *Candidates who attempted this part received the majority of the points.*

(d)

(i) Under the Markov property, for a policy in State 1 at time t, the probability of moving from State 1 to State 2 in any future time interval is independent of the history of the state process before time t.

This means, that (for example) the probability of a life dying in the next 6 months for a policyholder currently in State 1, is the same if they were diagnosed, say, in the last week as if they were diagnosed several years ago.

(ii) This is inconsistent with the spike in mortality observed after diagnosis, as this implies a higher mortality rate for lives recently arrived in State 2, compared with lives who have been in State 2 for a long time.

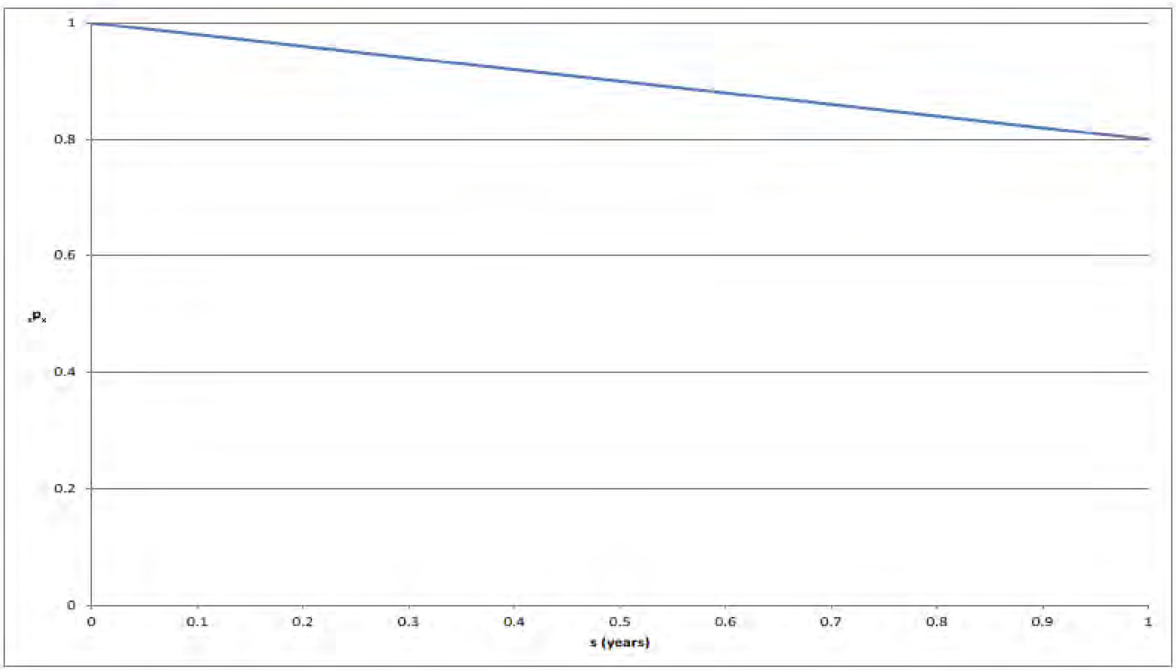
Graders' Comments: *Many candidates skipped this part.*

Question 17

(a) The symbol q_{xy} is the probability that at least one of two lives, currently age x and y, dies within one year.

Graders' Comments: *Part (a) was well done by almost all candidates. For full credit, candidates were required to specify the 1-year time period for the mortality probability.*

(b) The function is



Graders' Comments: *Most candidates did well on this part. Candidates who sketched a non-linear function for $_tp_x$, or who sketched a different probability, or who sketched a line from 1 to 0, received no credit for this part.*

(c)

$${}_{s}q_{xy} = 1 - {}_{s}p_x \cdot {}_{s}p_y \quad \text{(independence)}$$

$$= 1-(1-{}_{s}q_x)(1-{}_{s}q_y) = 1-(1-s\cdot q_x)(1-s\cdot q_y) \quad \text{(UDD)}$$

$$= s(q_x + q_y) - s^2 q_x q_y$$

Also $q_x + q_y = q_{xy} + q_{\overline{xy}}$ and $q_{\overline{xy}} = q_x q_y$

$$\Rightarrow {}_{s}q_{xy} = s(q_{xy} + q_{\overline{xy}}) - s^2 q_{\overline{xy}}$$

$${}_{s}q_{xy} = s(q_{xy}) + (s - s^2) q_{\overline{xy}}$$

$$\Rightarrow g(s) = s - s^2$$

Graders' Comments: *Performance on this part was mixed. Many candidates received full credit. A number of candidates made a reasonable start but could not derive the final result. Partial credit was awarded in these cases. The most common error was assuming that UDD applied to the joint life status.*

Question 18

(a) The future lifetimes of (x) and (y) are dependent, because the force of mortality for each is different depending on whether the other is alive or not, as $\mu^{01}_{x+t:y+t} \neq \mu^{23}_{x+t}$ and $\mu^{02}_{x+t:y+t} \neq \mu^{13}_{y+t}$. That means that the distribution of the time to death of (x) is different if (y) dies early than if (y) dies later (and vice versa), which means the future lifetime random variables T_x and T_y are dependent.

Graders' Comments: *The key point, is that the force of mortality for each life is different from the married state than from the widowed state. Many candidates wrongly claimed that the lives are independent because there is no common shock transition.*

(b) (i)

$$\frac{d}{dt}\,{}_t p^{00}_{xy} = -{}_t p^{00}_{xy}\left(\mu^{01}_{x+t:y+t} + \mu^{02}_{x+t:y+t}\right) \text{ with boundary condition of } {}_0 p^{00}_{xy} = 1$$

(ii)

$$\frac{d}{dr}\,{}_r p^{00}_{xy} = -{}_r p^{00}_{xy}\left(\mu^{01}_{x+r:y+r} + \mu^{02}_{x+r:y+r}\right) \Rightarrow \frac{1}{{}_r p^{00}_{xy}}\frac{d}{dr}\,{}_r p^{00}_{xy} = -\left(\mu^{01}_{x+r:y+r} + \mu^{02}_{x+r:y+r}\right)$$

$$\Rightarrow \frac{d}{dr}\ln\left({}_r p^{00}_{xy}\right) = -\left(\mu^{01}_{x+r:y+r} + \mu^{02}_{x+r:y+r}\right)$$

Integrate both sides from 0 to t

$$\int_0^t \frac{d}{dr}\ln\left({}_r p^{00}_{xy}\right)dr = \int_0^t -\left(\mu^{01}_{x+r:y+r} + \mu^{02}_{x+r:y+r}\right)dr \Rightarrow \ln\left({}_t p^{00}_{xy}\right) - \ln\left({}_0 p^{00}_{xy}\right) = \int_0^t -\left(\mu^{01}_{x+r:y+r} + \mu^{02}_{x+r:y+r}\right)dr$$

from the boundary condition, $\ln\left({}_0 p^{00}_{xy}\right) = 0,$ so

$$\ln\left({}_t p^{00}_{xy}\right) = \int_0^t -\left(\mu^{01}_{x+r:y+r} + \mu^{02}_{x+r:y+r}\right)dr \Rightarrow {}_t p^{00}_{xy} = \exp\left(-\left(\mu^{01}_{x+r:y+r} + \mu^{02}_{x+r:y+r}\right)dr\right)$$

Graders' Comments: *Most candidates scored partial credit for this part. Some candidates wrote down a few steps, but missed the key parts of the proof. An acceptable alternative approach was to plug the given solution into the Kolmogorov equation and demonstrate that the integral equation for ${}_t p^{00}_{xy}$ satisfies the Kolmogorov differential equation and the boundary condition. This approach was awarded full points if done correctly.*

(c) (i) At time 10, (x) is age 60 and (y) is age 65. The value at time 10 of the joint and reversionary annuities is

$$50,000\bar{a}^{00}_{60:65} + 30,000\bar{a}^{01}_{60:65} + 30,000\bar{a}^{02}_{60:65}$$

$$= (50,000)(8.8219) + (30,000)(1.3768) + (30,000)(3.0175) = 572,924$$

(ii) The EPV of the deferred annuity for the case where both lives survive 10 years uses the results from (i):

$$_{10}p^{00}_{50:55} \cdot v^{10} \cdot 572,924 = (0.86041)(0.61391)(572,924) = 302,627$$

The EPV of the deferred annuity for the case where (y) survives 10 years but (x) does not, is

$$_{10}p^{01}_{50:55} \cdot v^{10}(30,000\bar{a}^{11}_{65}) = (0.04835)(0.61391)(30,000)(10.1948) = 9078$$

The EPV of the deferred annuity for the case where (x) survives 10 years but (y) does not, is

$$_{10}p^{02}_{50:55} \cdot v^{10}(30,000\bar{a}^{22}_{60}) = (0.08628)(0.61391)(30,000)(11.8302) = 18,799$$

Let P denote the premium. Then the EPV of the benefit paid on the second death during the deferred period is

$$3P\bar{A}^{03}_{50:55:\overline{10|}} = 3P(0.003421) = (0.010263)P$$

$$\Rightarrow \quad P = 302,627 + 9078 + 18,799 + 0.010263P$$

$$\Rightarrow P = \frac{330,504}{0.989737} = 333,931$$

Graders' Comments: *Only the top few candidates achieved full credit for this part. Many candidates did not allow for the possibility that only one life would survive the deferred period.*

(d) (i)

$$_{10}V^{(0)} = 572,924 \qquad \text{from (c)(i)}$$

$$_{10}V^{(1)} = 30,000\bar{a}^{11}_{65} = (30,000)(10.1948) = 305,844$$

$$_{10}V^{(2)} = 30,000\bar{a}^{22}_{60} = (30,000)(11.8302) = 354,906$$

(ii) For $t \geq 10$ and $\delta = \log(1.05)$

$$\frac{d}{dt}\,{}_tV^{(0)} = \delta\,{}_tV^{(0)} - 50,000 - \mu^{01}_{50+t:55+t}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu^{02}_{50+t:55+t}\left({}_tV^{(2)} - {}_tV^{(0)}\right)$$

(iii) We need $\mu^{01}_{60:65} = A + Bc^{60} = 0.009076$ and $\mu^{02}_{60:65} = A + Bc^{65} = 0.015919$. Then

$$_{t+h}V^{(0)} \approx {}_tV^{(0)} + h\left(\delta_t V^{(0)} - 50,000 - \mu^{01}_{50+t:55+t}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu^{02}_{50+t:55+t}\left({}_tV^{(2)} - {}_tV^{(0)}\right)\right)$$

$$\approx 572,924 + 0.5\{(0.048790)(572,924) - 50,000 - (0.009076)(305,844 - 572,924) \\ - (0.015919)(354,906 - 572,924)\}$$

$$= 564,848$$

Graders' Comments: *As in (c), only the strongest candidates achieved full credit for this part. Many students correctly calculated the three policy calues, though some did not apply the correct annuity values ti the state-dependent polict values.*

Question 19

(a) The equation of value is

$$P\ddot{a}_{\overline{40:40}:\overline{20}|} = 100,000 A_{\overline{40:40}}$$

where $A_{\overline{40:40}} = 2A_{40} - A_{40:40} = 0.08157$

and $\ddot{a}_{\overline{40:40}:\overline{20}|} = 2\ddot{a}_{40:\overline{20}|} - \ddot{a}_{40:40:\overline{20}|} = 13.0842$

$\Rightarrow P = 623.42$

Graders' Comments: *Candidates did very well on this part, most of them achieving full credit.*

(b)

- **Common shock**: both lives may die at the same time, e.g. as a result of an accident.
- **Common lifestyle**: couples tend to share a common lifestyle, e.g. healthy (exercising) or unhealthy (smoking) habits.
- **Broken heart syndrome**: mortality may increase temporarily after the death of one's partner.

Graders' Comments: *Almost all candidates stated at least one valid reason, and most candidates received full credit for this part.*

(c)

(i) $A_{40:40} = 1 - d\,\ddot{a}_{40:40}$

$$\ddot{a}_{40:40} = \ddot{a}_{40:40:20} + {}_{20}E_{40:40}\ \ddot{a}^{\text{SULT}}_{60:60} = 17.6836$$

$$\Rightarrow A_{40:40} = 1 - \frac{0.05}{1.05}(17.6836) = 0.15792$$

(ii) $${}_{20}E_{40:40} = {}_{10}E_{40:40} \times {}_{10}E_{50:50} \Rightarrow {}_{10}E_{40:40} = \frac{0.35912}{0.59290} = 0.60570$$

(iii) $$\ddot{a}_{40:40:\overline{20}|} = \ddot{a}_{40:40:\overline{10}|} + {}_{10}E_{40:40}\ \ddot{a}_{50:50:\overline{10}|} \Rightarrow \ddot{a}_{50:50:\overline{10}|} = \frac{12.9254 - 8.0703}{0.60570} = 8.0157$$

Graders' Comments: *Performance on this part was mixed. Many candidates did not correctly incorporate the independent SULT functions applying after 20 years, and some did not identify or utilize the relationship between the A and ä functions.*

(d) We now have

$$A_{\overline{40:40}} = 2A_{40} - A_{40:40} = 0.08420$$

$$\ddot{a}_{\overline{40:40}:\overline{20}|} = 2\ddot{a}_{40:\overline{20}|} - \ddot{a}_{40:40:\overline{20}|} = 13.0616$$

$\Rightarrow P = 644.62$

(e)

$(i)\quad \mathrm{E}\left[{}_{10}L \mid \text{only 1 life alive} \right] = 100{,}000 A_{50} - P\ddot{a}_{50:\overline{10}|}$

$= 13{,}738.6$

$(ii)\ \mathrm{E}\left[{}_{10}L \mid \text{both alive} \right] = 100{,}000 A_{\overline{50:50}} - P\ddot{a}_{\overline{50:50}:\overline{10}|} = 8{,}223.3$

$(iii)\quad \mathrm{E}\left[{}_{10}L \mid \text{at least 1 alive} \right]$

$$= \frac{\mathrm{E}\left[{}_{10}L \mid \text{only 1 life alive} \right] \times \Pr\left[\text{only 1 life alive}\right] + \mathrm{E}\left[{}_{10}L \mid \text{both alive} \right] \times \Pr\left[\text{both alive}\right]}{\Pr\left[\text{at least one alive}\right]}$$

$\Pr\left[\text{Both alive}\right] = {}_{10}p_{40:40} = 0.9866$

$\Pr[\text{only 1 alive}] = {}_{10}p_{\overline{40:40}} - {}_{10}p_{40:40} = 0.0114$

$\Pr\left[\text{At least 1 alive}\right] = {}_{10}p_{\overline{40:40}} = 0.998$

$\Rightarrow \quad \mathrm{E}\left[{}_{10}L \mid \text{at least 1 alive} \right] = 8{,}286.2$

Graders' Comments: *Most candidates found this part challenging. Many received at least partial credit for (i) and (ii). Only a few candidates received full credit for part (iii).*

(f) The policy value used should correspond most closely with the information available. The insurance company knows that no claim has been made, so it knows that both lives have not died, but it does not know whether one or both lives are alive. Correspondingly, it should hold the reserve based on the policy value in (iii), which is suitable for the information that at least one life has survived. For each individual policy, this will be the wrong number – the correct number will either be the value in (i) or the value in (ii), but overall the value in (iii) will be most accurate in aggregate.

Graders' Comments: *Only the stronger candidates correctly identified the most suitable value for the policy value and provided an adequate explanation.*

Question 20

(a) Common shock:
Simultaneous deaths represented by transitions from State 0 to State 4.
Broken heart syndrome:
Mortality of surviving life is higher than individual mortality when both are alive, $\mu^{13} > \mu^{02}$ and $\mu^{23} > \mu^{01}$.

Graders' Comments: *Performance on this part was mixed. Many candidates identified only one way that dependency is incorporated in the model.*

(b)

(i) $${}_{10}p_{50}^{23} = 1 - \exp\left(-\int_0^{10} 1.05\mu_{50+t}^{*}\,dt\right) = 1 - \exp\left(-\int_0^{10} \mu_{50+t}^{*}\,dt\right)^{1.05}$$
$$= 1 - \left({}_{10}p_{50}\right)^{1.05} \quad \text{where } {}_{10}p_{50} \text{ is from the SULT}$$
$$= 0.020678$$

(ii) $${}_{10}p_{40:50}^{00} = \exp\left(-\int_0^{10} \mu_{40+t:50+t}^{01} + \mu_{40+t:50+t}^{02} + \mu_{40+t:50+t}^{04}\,dt\right)$$
$$= \exp\left(-\int_0^{10} \left(\mu_{50+t}^{*} - 0.0005\right) + \left(\mu_{40+t}^{*} - 0.0005\right) + 0.0005\,dt\right)$$
$$= e^{0.005}\ {}_{10}p_{40}\ {}_{10}p_{50} = 0.977654$$

Graders' Comments: *Only the most well-prepared candidates achieved full or nearly full credit on this part. Partial credit was awarded to candidates who showed some understanding of the question by writing down some formulas relevant to the calculation of these probabilities of transition..*

(c) Note that we must use the given information, that $\bar{a}_{40:50:\overline{10|}}^{00} = 7.8487$ and $\bar{A}_{40:50:\overline{10|}}^{03} = 0.00789$. The equation of value is

$$P\bar{a}_{40:50:\overline{10|}}^{00} = 100{,}000\bar{A}_{40:50:\overline{10|}}^{03} + 300{,}000\bar{A}_{40:50:\overline{10|}}^{04}$$
$$\bar{A}_{40:50:\overline{10|}}^{04} = \int_0^{10} {}_tp_{40:50}^{00}\,\mu_{40+t:50+t}^{04}\,e^{-\delta t}\,dt = 0.0005\int_0^{10} {}_tp_{40:50}^{00}\,\mu_{40+t:50+t}^{04}\,e^{-\delta t}\,dt$$
$$= 0.0005\,\bar{a}_{40:50:\overline{10|}}^{00} = 0.00392435$$
$$\Rightarrow P = 250.53$$

Graders' Comments: *Most candidates were able to write down the formula needed to calculate the premium. Some candidates were also able to write down a correct expression for* $\bar{A}_{40:50:\overline{10|}}^{04}$ *but only a few recognized that it could be calculated from the given value for* $\bar{a}_{40:50:\overline{10|}}^{00}$.

(d) (i) $\bar{a}^{00}_{x:y:\overline{10}|}$: Stays the **same**.

The transition intensities leaving State 0 do not change and it is impossible to return to State 0 after leaving it.

(ii) $\bar{A}^{03}_{x:y:\overline{10}|}$: Would be **higher**.

If (y) dies first, no impact. If (x) dies first, (y) is likely to die sooner which will increase the EPV of the insurance benefit.

(iii) $\bar{a}_{x|y}$: This corresponds to $\bar{a}^{02}_{x:y}$ in this model. Would be **lower**.

This annuity is payable while (y) is alive after the death of (x). Since in this case, (y) is likely to die sooner, the EPV of the annuity payments will decrease.

(iv) Premium: Would be **higher**.

Since $\bar{a}^{00}_{x:y:\overline{10}|}$ and $\bar{A}^{04}_{x:y:\overline{10}|}$ would stay the same but $\bar{A}^{03}_{40:50:\overline{10}|}$ would increase, the premium in (c) would increase.

Graders' Comments: *Most candidates received partial credit for correctly justifying the impact of the change on some of the four given actuarial functions. A small number of candidates adequately justified the impact of the change on all four functions.*

Question 21

(a) Due to independence:

$$ {}_{10}p_{70:70} = ({}_{10}p_{70})^2 = \left(\frac{l_{80}}{l_{70}}\right)^2 = \left(\frac{75,657.2}{91,082.4}\right)^2 = 0.689972 $$

(b) $E[L_0] = 100{,}000A_{70:70:\overline{10}|} + 400 - G(0.98\,\ddot{a}_{70:70:\overline{10}|} - 0.48)$ where

$\ddot{a}_{70:70:\overline{10}|} = 7.2329$ (from SULT)

$$A_{70:70:\overline{10}|} = 1 - d\ddot{a}_{70:70:\overline{10}|} = 1 - (0.04762)(7.2329) = 0.65557$$

Then $E[L_0] = 65{,}557 + 400 - 10{,}658\,(0.98(7.2329) - 0.48)$

$= -4{,}474$

(c)

(i) $$\begin{aligned} {}^2A_{70:70:\overline{10}|} &= {}^2A_{70:70} + {}_{10}p_{70:70}\ v^{20}\,(1 - {}^2A_{80:80}) \\ &= 0.30743 + 0.689972\,(1.05)^{-20}\,(1 - 0.50165) \\ &= 0.437 \end{aligned}$$

(ii) $$L_0 = 100{,}000\, v^{min(K_{70:70}+1,10)} + 400 - G\left(0.98\,\ddot{a}_{\overline{min(K_{70:70}+1,10)}|} - 0.48\right)$$

$$L_0 = 100{,}000v^{\min(K_{70:70}+1,10)} + 400 - G\left((0.98)\left(\frac{1-v^{\min(K_{70:70}+1,10)}}{d}\right) - 0.48\right)$$

$$L_0 = \left(100{,}000 + \frac{0.98\,G}{d}\right) v^{min(K_{70:70}+1,10)} + 400 - G\left(\frac{0.98}{d} - 0.48\right)$$

$$Var[L_0] = \left(100{,}000 + \frac{0.98\,G}{d}\right)^2 Var\left[v^{min(K_{70:70}+1,10)}\right]$$

$$= \left(100{,}000 + \frac{0.98\,G}{d}\right)^2 \left({}^2A_{70:70:\overline{10}|} - (A_{70:70:\overline{10}|})^2\right) = 737{,}080{,}000$$

$SD[L_0] = 27{,}150$

(d)

Let $L_{0,i}$ be the loss at issue for policy i, i=1,2,…,100.

The aggregate loss at issue is $L = \sum_{i=1}^{100} L_{0,i}$.

Then, $E[L\,] = 100\,E[L_0] = -447{,}400$

$Var[L\,] = 100Var[L_0]$

$SD[L\,] = 10SD[L_0] = 271{,}500$

$$
\begin{aligned}
P[L > 0] &\approx 1 - \Phi\left(\frac{0-(-447{,}400)}{271{,}500}\right) \\
&\approx 1 - \Phi(1.648) \\
&\approx 0.05
\end{aligned}
$$

(e)

The total profit realized in this sub-portfolio will be **greater** than its expected value.

Justification:

There were two death benefits paid during the term of the contract and 8 endowment benefits paid at maturity (time 10).

The expected number of death benefits in the sub-portfolio was 10 $(1 - {}_{10}p_{70:70})$ which represents 3.1 death benefits.

The EPV of the 2 death benefits was lower than expected.

Or

The mortality experience in this sub-portfolio was better than expected and the death benefits were paid relatively late in the term of the policy (at times 6 and 9).

Consequently, the EPV of the benefits was lower than expected.

Graders' Comments: *The majority of candidates did well on parts (a) to (d) with many receiving full credit. Part (e) proved much more challenging, with few candidates receiving full credit.*

Question 22

(a)

The probability is:

$$\Pr[1 \text{ alive}] = 2 \times {}_{20}p_{50}(1 - {}_{20}p_{50})$$

$${}_{20}p_{50} = \frac{l_{70}}{l_{50}} = \frac{91{,}082.4}{98{,}576.4} = 0.92398$$

$$\Rightarrow (2)(0.92398)(1-0.92398) = 0.1405$$

(b)

$$\ddot{a}^{(2)}_{\overline{50:50}:\overline{20|}} = 2\ddot{a}^{(2)}_{50:\overline{20|}} - \ddot{a}^{(2)}_{50:50:\overline{20|}}$$

$$\ddot{a}^{(2)}_{50:\overline{20|}} = \ddot{a}_{50:\overline{20|}} - \frac{1}{4}\left(1 - {}_{20}E_{50}\right) = 12.8428 - 0.25(1-0.34824) = 12.6799$$

$$\ddot{a}^{(2)}_{50:50:\overline{20|}} = \ddot{a}^{(2)}_{50:50} - {}_{20}E_{50} \times \ddot{a}^{(2)}_{70:70} = \left(\ddot{a}_{50:50} - \frac{1}{4}\right) - \left({}_{20}p_{50}\right)^2 (1.05)^{-20}\left(\ddot{a}_{70:70} - \frac{1}{4}\right)$$

$$(15.8195 - 0.25) - (0.92398)^2(1.05)^{-20}(9.9774 - 0.25) = 12.4396$$

$$\Rightarrow \ddot{a}^{(2)}_{\overline{50:50}:\overline{20|}} = (2)(12.6799) - 12.4396 = 12.9202$$

Graders' Comments: *Many students attempted this part, and most of them received partial credit for correct intermediate steps. The most common errors were (i) incorrect application of Woolhouse's formula; (ii) using a joint endowment instead of a last survivor when calculating the adjusted last survivor term annuity; (iii) Incorrect Woolhouse coefficients (11/24 or ½ instead of 1/4)*

(c)

EPV Benefits:

$$100{,}000\, A_{\overline{50:50}} = 100{,}000(2A_{50} - A_{50:50}) = 100{,}000(2 \times 0.18931 - 0.24669) = 13{,}193$$

EPV Prems – Exp:

$$G\left\{\left(\ddot{a}^{(2)}_{\overline{50:50}:\overline{20|}} + 0.1405 v^{20}\ddot{a}_{70} + {}_{20}p_{50:50} v^{20}\ddot{a}_{\overline{70:70}}\right) \times 0.9 - 0.7\right\}$$

$$\ddot{a}_{\overline{70:70}} = 2 \times \ddot{a}_{70} - \ddot{a}_{70:70} = (2)(12.0083) - 9.9774 = 14.0392$$

$$\Rightarrow \text{ EPV Prems } - \text{Exp} = G\left\{\left(12.9202 + 0.1405 \times v^{20} \times 12.0083 + (0.92398)^2 (v^{20})(14.0392)\right)0.9 - 0.7\right\}$$

$$= G\{0.9 \times 18.0734 - 0.7\} = 15.5661G$$

$$\Rightarrow G = \frac{13{,}193}{15.5661} \quad = 847.55$$

Graders' Comments: *Not many candidates received full credit, but most of those that attempted this part received the majority of points for correctly setting up the EPV formulas. Common mistakes included :*

- *Incorrect expense payment (60% vs. 70%)*
- *Missing the annuity portion due when exactly 1 person is alive*
- *Multiplying the first annuity term by 2 (incorrect treatment of semi-annual premium payment)*

(d)

(i) Let ${}_{20}V^{(2)}$ denote the reserve conditional on both surviving:

$${}_{20}V^{(2)} = 100000A_{\overline{70:70}} - (0.9)(847.55)\ddot{a}_{\overline{70:70}}$$

$$A_{\overline{70:70}} = 1 - d\ddot{a}_{\overline{70:70}} = 1 - \left(\frac{0.05}{1.05}\right)(14.0392) = 0.33147$$

$$\text{or } (2)A_{70} - A_{70:70} = (2)(0.42818) - 0.52488 = 0.33148$$

$$\Rightarrow {}_{20}V^{(2)} = 22{,}437.6$$

(ii) Let ${}_{20}V^{(1)}$ denote the reserve conditional on exactly one surviving:

$$\begin{aligned} {}_{20}V^{(1)} &= 100{,}000A_{70} - (0.9)(847.55)\ddot{a}_{70} \\ &= (100{,}000)(0.42818) - (0.9)(847.55)(12.0083) \\ &= 33{,}658 \end{aligned}$$

Graders' Comments: *Candidates who attempted this part generally received the majority of the credit. The most common mistake was neglecting the 90% coefficient on the annuity portion of the calculation.*

(e) $\left({}_{19.5}V^{(2)} + 0.9 \times G/2\right)(1.05)^{0.5} = \underbrace{\left({}_{0.5}q_{69.5}\right)^2 \times 100{,}000}_{\text{Both Insureds Die}} + \underbrace{\left({}_{0.5}p_{69.5}\right)^2 \; {}_{20}V^{(2)}}_{\text{Both Insureds Live}}$

$$+ \underbrace{2\left({}_{0.5}q_{69.5} \; {}_{0.5}p_{69.5}\right) {}_{20}V^{(1)}}_{\substack{\text{One Insured Lives and One Dies} \\ \text{The (2) in front is because either could die.}}}$$

Also, ${}_{0.5}p_{69.5} = \left(p_{69}\right)^{0.5} = (1 - 0.009294)^{0.5} = 0.99534$

$\Rightarrow {}_{19.5}V^{(2)} = 21{,}619$

Graders' Comments: *Not many candidates attempted this part, and very few received full credit. When they attempted to solve for the survival probability, students generally performed the calculation correctly, the majority of mistakes came from the recursion equation.*

Question 23

(a)

$\bar{a}_{\mathrm{xy}}^{01}$ is the actuarial value of a (whole-life) annuity paid to (y) starting on the death of (x), payable continuously at a *rate* of 1 per year.

Or

A reversionary annuity paid to (y) starting on the death of (x), payable continuously at a *rate* of 1 per year.

Graders' Comments:_*Most candidates did very well. Some candidates didn't mention "continuous", and a few candidates only provided a description that was not specific to the given joint life model (e.g., "a continuous annuity payable at a rate of 1 per year while in state 1 given that it is issued in state 0")*

(b)

$$\frac{d}{dt}\ {}_tp_{xy}^{00} = -{}_tp_{xy}^{00}\left(\mu_{x+t}^{01} + \mu_{y+t}^{02}\right) ; \qquad {}_0p_{xy}^{00} = 1$$

And

$$\frac{d}{dt}\ {}_tp_{xy}^{01} = {}_tp_{xy}^{00}\,\mu_{x+t}^{01} - {}_tp_{xy}^{01}\,\mu_{y+t}^{13} ; \qquad {}_0p_{xy}^{01} = 0$$

Graders' Comments: *Candidates did well on this part. A common error was incorrect or missing subscripts (i.e., ages). A few candidates didn't provide the boundary conditions.*

(c)

We have

$$\bar{a}_{xy}^{00} = \int_0^\infty {}_tp_{xy}^{00}\ e^{-\delta t}\, dt = \int_0^\infty g(t)\, dt$$

and

$$\ddot{a}_{xy}^{00} = \sum_{k=0}^{\infty} {}_kp_{xy}^{00}\ e^{-\delta k} = \sum_{k=0}^{\infty} g(k)$$

i.e. $\bar{a}$ is the integral, $\ddot{a}$ the sum

where

$$g(t) = {}_tp_{xy}^{00}\ e^{-\delta t}\ , \qquad g(0) = 1.$$

The derivative of g(t) using the product rule is

$$g'(t) = \left(\frac{d}{dt}\,{}_tp_{xy}^{00}\right)e^{-\delta t} + {}_tp_{xy}^{00}\left(\frac{d}{dt}e^{-\delta t}\right)$$

$$= -{}_tp_{xy}^{00}\left(\mu_{x+t}^{01} + \mu_{y+t}^{02}\right)e^{-\delta t} + {}_tp_{xy}^{00}\left(-\delta e^{-\delta t}\right).$$

So, $g'(0) = -1 \cdot \left(\mu_x^{01} + \mu_y^{02}\right) \cdot 1 + 1 \cdot (-\delta \cdot 1) = -\left(\mu_x^{01} + \mu_y^{02} + \delta\right)$

The approximation

$$\int_0^\infty g(t)\, dt \approx \sum_{k=0}^{\infty} g(k) - \frac{1}{2}g(0) + \frac{1}{12}g'(0)$$

then gives

$$\bar{a}_{xy}^{00} \approx \ddot{a}_{xy}^{00} - \frac{1}{2} - \frac{1}{12}\left(\mu_x^{01} + \mu_y^{02} + \delta\right).$$

Graders' Comments: *Candidates did fairly well, but not many received full credit. The most common issue was the derivative, g'(t). A number of candidates mistakenly assumed constant forces of mortality, receiving little credit.*

(d)

Let B = annual annuity payment rate

$1{,}000{,}000 = B\ \bar{a}_{\overline{70:70}} = B\left(\bar{a}_{70:70}^{00} + \bar{a}_{70:70}^{01} + \bar{a}_{70:70}^{02}\right)$

Since $\bar{a}_{70:70}^{01} = \bar{a}_{70:70}^{02} = \bar{a}_{70|70} = 2.0317$

$1{,}000{,}000 = B\ \bar{a}_{\overline{70:70}} = B\left(\bar{a}_{70:70}^{00} + 2\bar{a}_{70|70}\right)$

Using the approximation in (c),

$$\bar{a}^{00}_{70:70} \approx \ddot{a}^{00}_{70:70} - \frac{1}{2} - \frac{1}{12}(2\mu_{70} + \delta)$$

$$= \ddot{a}^{SULT}_{70:70} - \frac{1}{2} - \frac{1}{12}(2(0.009881) + 0.04879) = 9.4717$$

⇨ $B = \frac{1{,}000{,}000}{9.4717+2(2.0317)} = \frac{1{,}000{,}000}{13.5351} = 73{,}881.98$

Graders' Comments: *Candidates did fairly well on this part. A few candidates mistakenly applied the EMW approximation in Part c) for a last-survivor annuity. Note that the formula for g(t) is different for different cases, the EMW approximation for a last-survivor annuity would be different from c).*

Question 24

(a)

$$200{,}000\, A_{50:60:\overline{10|}} = P\, \ddot{a}_{50:60:\overline{10|}}$$
$$\ddot{a}_{50:60:\overline{10|}} = 7.9044$$
$$A_{50:60:\overline{10|}} = 1 - d\, \ddot{a}_{50:60:\overline{10|}} = 1 - \frac{0.05}{1.05}(7.9044) = 0.6236$$

$$P = \frac{(200,000)A_{50:60:\overline{10|}}}{\ddot{a}_{50:60:\overline{10|}}} = \frac{(200,000)(0.6236)}{7.9044} = 15,778.55$$

(b)

$${}^2A_{50:60:\overline{10|}} = {}^2A_{50:60} + v^{20}\,{}_{10}p_{50}\ {}_{10}p_{60}\,(1 - {}^2A_{60:70})$$

$$= {}^2A_{50:60} + {}_{10}E_{50}\ {}_{10}E_{60}\,(1 - {}^2A_{60:70})$$

$$= 0.3908335$$

Graders' Comments*: Candidates did well on these parts. Common errors were missing the endowment part of the equation or using wrong discount factors for the endowment.*

(c)

$$Var[L] = \left(S + \frac{P}{d}\right)^2 \left({}^2A_{50:60:\overline{10|}} - (A_{50:60:\overline{10|}})^2\right)$$

$$= (531{,}349.55)^2(0.390834 - 0.6236^2) = (531{,}349.55)^2(0.001957)$$

$$= 552{,}524{,}514.1 = 23{,}505.84^2$$

Standard Deviation of L = 23,505.84

Graders' Comments: *Overall, candidates did well on this part also. Many candidates have this formula memorized. Partial credit was awarded if there was an attempt to derive the formula, if it wasn't memorized.*

(d)

$$G = \frac{S\ A_{50:60:\overline{10|}}}{(0.9)\ddot{a}_{50:60:\overline{10|}} - 0.15} = \frac{124{,}720}{6.96396} = 17{,}909.35$$

***Graders' Comments**: Again, candidates did well on this part. The most common error was the double counting first year expense (using a separate 0.25 instead and using .15)*

(e)

(i) $G^* = \frac{S\ {}^2A_{50:60:\overline{10}|}}{(0.9)\ {}^2\ddot{a}_{50:60:\overline{10}|} - 0.15} = \frac{200{,}000(0.390834)}{5.747024} = 13{,}601.27$

$${}^2\ddot{a}_{50:60:\overline{10}|} = \frac{1 - {}^2A_{50:60:\overline{10}|}}{d^*} = 6.552249$$

$$d^* = 0.1025/1.1025 = 0.0929705$$

A reduction of (17,909.35 − 13,601.27)/17,909.35 = 0.2405 or 24.05%

(ii) The percentage reduction would be **less** for a 10-year term insurance. The endowment insurance also pays a benefit in case of survival to time 10. The percentage reduction in the EPV of the survival benefit will be larger than the percentage reduction in the EPV of the death benefit (which is paid early, on average, and is therefore relatively less affected by a change in the interest rate).

***Graders' Comments:** For Part (i) the most common error was not calculating the correct discount rate or calculating the net premium instead of the gross premium. For Part (ii), many candidates did not really address the question answered and instead commented on expenses or joint life versus single life as opposed to the difference between term and endowment.*

Question 25

(a)

The future lifetimes are <u>not</u> independent. The mortality of each life is different when their partner is alive than it is when their partner is dead. This shows that the future lifetime of each partner impacts the future lifetime of the other. OR: A necessary condition for independence is that $\mu_y^{01} = \mu_y^{23}$ and $\mu_x^{02} = \mu_x^{13}$. As this is not the case, the future lifetimes are not independent.

Graders' Comments: *Most candidates who correctly identified dependence gave good reasoning. Many candidates failed to recognize dependence, assuming that no direct transitions from State 0 to State 3 implied independence.*

(b)

$$ {}_t p_{xy}^{00} = \exp\left(-\int_0^t \mu_{x+r:y+r}^{01} + \mu_{x+r:y+r}^{02}\, dr\right) = \exp\left(-\int_0^t \mu_{y+r}^{s} + \mu_{x+r}^{s} - 0.005\, dr\right) $$

$$ = e^{0.005t}\ {}_t p_x^s\ {}_t p_y^s \quad \text{==>}\ \lambda = 0.005 $$

Graders' Comments: *This is a "show that" question, where candidates need to be very precise about their derivation. Overall candidates did well on this part. However, many candidates assumed constant forces of mortality, which is incorrect and unnecessary.*

(c)

$$ \bar{a}_{x:y}^{00} = \int_0^\infty {}_t p_{x:y}^{00}\, e^{-\delta t} dt = \int_0^\infty {}_t p_x^s\ {}_t p_y^s\, e^{\lambda t} e^{-\delta t} dt = \bar{a}_{x:y}^{s}\Big|_{\delta^*} \quad \text{where } \delta^* = \delta - \lambda = 0.035 $$

$$ \Rightarrow \bar{a}_{40:50}^{00} = 19.3199 $$

Graders' Comments: *The point of this part is to get candidates to work out the correct interest rate to use. For full credit, they need to justify their choice. Many candidates overcomplicated the question, spending time integrating, instead of using the table (this was especially true for candidates who assumed constant forces of mortality).*

(d)

(i) $\bar{A}_{x:y}^{01} + \bar{A}_{x:y}^{02} = \int_0^\infty {}_t p_{x:y}^{00} \left(\mu_{y+t}^{01} + \mu_{x+t}^{02}\right) e^{-\delta t} dt$

(ii) $\bar{A}^{01}_{x:y} + \bar{A}^{02}_{x:y} = \int_0^\infty {}_tp^s_x \; {}_tp^s_y \left(\mu^s_{y+t} + \mu^s_{x+t} - 0.005\right) e^{-\delta^* t} dt$

$$= \bar{A}^s_{x:y\,|\delta*} - 0.005\,\bar{a}^s_{x:y\,|\delta*} = 1 - \delta^*\bar{a}^s_{x:y\,|\delta*} - 0.005\,\bar{a}^s_{x:y\,|\delta*}$$

$$= 1 - \delta\bar{a}^s_{x:y\,|\delta*} = 1 - (0.04)(19.3199) = 0.227204$$

Graders' Comments: *Most candidates either skipped this part or assumed constant forces of mortality.*

(e)

$$P = \frac{500,000\left(\bar{A}^{01}_{x:y} + \bar{A}^{02}_{x:y}\right)}{\bar{a}^{00}_{x:y}} = \frac{(500,000)(0.227204)}{19.3199} = 5,880.1$$

(f)

$$X = \frac{500,000}{\bar{a}^{11}_{50}} = \frac{500,000}{\bar{a}^s_{x:y\,|\delta**=0.0425}} = \frac{500,000}{17.9917} = 27,791$$

Graders' Comments: *Parts (e) and (f) were quite straightforward, relying on given answers from earlier parts. However, many candidates who struggled with part (d) skipped these parts.*

Question 26

(a)

$$\text{EPV Benefits} = (100,000)A_{45} = (100,000)(0.15161) = 15,161$$
$$\text{EPV Premiums} = P\ddot{a}_{45} = 17.8162P$$
$$\text{EPV Expenses} = 20\ddot{a}_{45} + 80 + 200A_{45} + 0.1P\ddot{a}_{45} + 0.65P$$
$$= (20)(17.8162) + 80 + (200)(0.15161) + [(0.1)(17.8162) + 0.65]P = 466.646 + 2.43162P$$

$$P = \frac{15,161 + 466.646}{17.8162 - 2.43162} = 1015.80$$

(b)

$$_1V = (100,200)A_{46} - 0.9P\ddot{a}_{46} + 20\ddot{a}_{46}$$
$$= (100,200)(0.15854) - [(0.9)(1015.80) - 20](17.6706) = 84.30$$

Graders' Comments: *Candidates could answer using the prospective approach or the recursion approach.. The most common errors involved incorrect treatment of expenses. Some candidates used the actual interest rate (1.07) instead of the pricing rate (1.05), which led to partial credit if the rest of the solution was correct.*

(c)

Interest Gain:

$$(10,000)(0.07 - 0.05)[P - (0.75P + 100)] = 30,790$$

Expense Gain:

$$(10,000)[(0.75P + 100) - (0.75P + 105)](1.07) = -53,500$$

Mortality Gain: 0

Commentary:

Candidates were expected to indicate clearly whether the amount calculated is a gain or a loss. A few candidates calculated only the total gain/loss; this received small partial credit. One relatively common mistake was to use a factor of .07 instead of 1.07 when calculating the expense gain and loss.

(d)

Interest: No gain or loss, as experience matches the assumption.
Expenses: Loss, as settlement expenses exceed the assumption.
Mortality: Expected deaths □ 10000 $p_{45}\, q_{46}$ □ 8.4 . There will be a loss, as actual deaths exceed expected deaths

Graders' Comments: *A few candidates compared the experience of year 2 with year 1. For example, "interest in year 2 was lower than in year 1 so there was a loss due to interest". This did not directly answer the question and did not receive full credit. A few candidates proposed that there would be a gain due to mortality because there were more deaths than expected.*

Question 27

(a)

$$\Pr_2 = ({}_1V + P - E)(1+i_2) - q_{51}S - p_{51}({}_2V)$$
$$= (400+1100-55)(1.02) - (0.00642)(100,000) - (0.99358)(800) = 37.04$$

(b)

t	${}_{t-1}V$	P	E_t	I_t	EDB_t	EMB	E_tV	$\Pr_t$
0			155					-155.00
1	0	1100	55	10.45	592		397.63	65.82
2	400	1100	55	28.90	642		794.86	37.04
3	800	1100	55	55.35	697	1092.33	0.00	111.02

The profit vector is the final column.

P is the premium

E_t denotes expenses

I_t denotes interest on funds in year t

$EDB_t = 100,000 q_{50+t-1}$

In year 3, $EMB_3 = p_{52} \times 1100$

$E_tV = p_{50+t-1}\ {}_tV$

Graders' Comments: *Many candidates received full credit for this part, and a larger number received partial credit. The most common errors included (i) ignoring the maturity benefit and (ii) incorrect use of probabilities in E_tV. It is not necessary for candidates to define all terms, but it can be helpful when graders are considering partial credit, especially if the values are calculated in Excel.*

(c) The profit signature at t is Π_t where:

$\Pi_0 = \Pr_0$ and $\Pi_t = {}_{t-1}p_{50} \cdot \Pr_t$

$$\Rightarrow (\Pi_0, \Pi_1, \Pi_2, \Pi_3) = (-155.00,\ 65.82,\ 36.82,\ 109.65)$$

$$\Rightarrow NPV = \sum_{k=0}^{3} \Pi_k \cdot v_{14\%}^k = 5.08$$

Graders' Comments: *Most candidates did this part well. The most common errors were incorrect survival probabilities when calculating the profit signature from the provit vector, and incorrect discounting periods for the NPV.*

(d) The IRR of B is j where $155 = 210v_j^3 \Rightarrow j = 10.65\%$.

and the IRR of A is greater than 14%, because the NPV at 14% is positive.
Hence IRR of B is less than IRR of A.

Product C has lower reserves in year 1,which allow an earlier release of surplus compared to Product A, which give a higher NPV than A at an 14% hurdle rate, but does not necessarily mean that C has a higher IRR.

The lower reserve in year 1 results in the following profit signature for C:
(-155.00, 165.23, -64.58, 109.65)

We note that the NPV of A is a little larger than 14%, because the NPV at 14% is close to zero. Calculating the NPV for A and C at 16% gives -0.65 for A and 9.7 for C. Hence, the IRR for C is greater than 16% , and for A is less than 16%.

That is $\Rightarrow IRR(B) < IRR(A) < IRR(C)$

Graders' Comments: *Many candidates evaluated the IRR for all three cases, presumably using the financial functions on the BA2 calculator. This was awarded full credit if correct. However, it was not necessary to determine the IRR to answer the question.*

Candidates who demonstrated understanding of the relationship between the release of surplus and the return to the insurer gained partial credit, even if the justification was incomplete. No credit was awarded for the IRR ordering if there was no accompanying explanation or justification

Question 28

(a) ${}_2p_{60}^{01} = p_{60}^{00}\, p_{61}^{01} + p_{60}^{01}\, p_{61}^{11}$

$= (0.9)(0.05) + (0.05)(0.2) = 0.055$

Graders' Comments:

1. *This part was done correctly by almost all candidates.*
2. *For those who did not receive full credit, two common errors were to use an incorrect annual probability of transition and to calculate only one of two terms.*

(b)

(i) ${}_2V^{(0)} = 50{,}000\, A_{62:\overline{8|}}^{02} + 150\, \ddot{a}_{62:\overline{8|}}^{00} + 150\, \ddot{a}_{62:\overline{8|}}^{01} - 5{,}000\, \ddot{a}_{62:\overline{8|}}^{00}$
$= 50{,}000(0.46667) + 150(4.7328) + 150(0.2533) - 5{,}000(4.7328)$
$= 23{,}333.5 + 709.92 + 37.995 - 23{,}664.0 = 417.415$

(ii) ${}_2V^{(1)} = 50{,}000\, A_{62:\overline{8|}}^{12} + 150\, \ddot{a}_{62:\overline{8|}}^{10} + 150\, \ddot{a}_{62:\overline{8|}}^{11} - 5{,}000\, \ddot{a}_{62:\overline{8|}}^{10}$
$= 50{,}000(0.4968) + 150(3.334 + 1.406) - 5{,}000(3.334)$
$= 24{,}840 + 711 - 16{,}670 = 8{,}881$

(iii) $\left({}_2V^{(0)} + 5{,}000 - 150\right)1.06 = p_{62}^{02}\,(50{,}000) + p_{62}^{00}\; {}_3V^{(0)} + p_{62}^{01}\; {}_3V^{(1)}$
$(417.415+4850)1.06=0.07(50{,}000)+0.88(1788) +0.05\; {}_3V^{(1)}$

$${}_3V^{(1)} = \frac{(5583.46 - 3500 - 1573.44)}{0.05} = 10{,}200.40$$

<u>Alternatively</u>

$\left({}_2V^{(1)} + 0 - 150\right)1.06 = p_{62}^{12}\,(50{,}000) + p_{62}^{10}\; {}_3V^{(0)} + p_{62}^{11}\; {}_3V^{(1)}$
$(8881-150)1.06=0.12(50{,}000)+0.68(1788)+0.20\; {}_3V^{(1)}$

$${}_3V^{(1)} = \frac{(9254.86 - 6000 - 1215.84)}{0.2} = 10{,}195.10$$

Graders' Comments: *Parts (i) and (ii) were done correctly by most candidates; for those who did not receive full credit, a common error was to ignore maintenance expenses. Although most candidates recognized that a recursive formula was needed to answer part (iii), setting it up correctly proved to be challenging for many candidates.*

(c)

(i) $Pr_3^{(0)} = \left({}_2V^{(0)} + P - 60\right)1.057 - p_{62}^{00}\,{}_3V^{(0)} - p_{62}^{01}\,{}_3V^{(1)} - p_{62}^{02}(50{,}000)$
$= (417.415 + 5000 - 60)1.057 - (0.88)(1788) - (0.05)(10{,}200.4) - (0.07)(50{,}000)$
$= 79.328$

$Pr_3^{(1)} = \left({}_2V^{(1)} - 60\right)1.057 - p_{62}^{10}\,{}_3V^{(0)} - p_{62}^{11}\,{}_3V^{(1)} - p_{62}^{12}(50{,}000)$
$= (8881 - 60)1.057 - (0.68)(1788) - (0.2)(10{,}200.4) - (0.12)(50{,}000)$
$= 67.877$

${}_2p_{60}^{01} = 0.055$ from (a)
${}_2p_{60}^{00} = (0.9)(0.89) + (0.05)(0.69) = 0.8355$
$\pi_3 = {}_2p_{60}^{00}\, Pr_3^{(0)} + {}_2p_{60}^{01}\, Pr_3^{(1)}$
$= (0.8355)(79.328) + (0.055)(67.877) = 70.012$

(ii) $NPV_t = NPV_{t-1} + \pi_t\, v^t = \sum_{k=0}^{t} \pi_k\, v^k; \quad NPV_0 = \pi_0; \; v = 1/1.08$

t	π_t	NPV_t
0	-200.00	-200.00
1	84.74	-121.54
2	80.35	- 52.65
3	70.01	2.93

Alternatively, with $\pi_3 = 70.291$, we get $NPV_3 = 3.150$
The Discounted Payback Period (DPP) is 3 years.

Graders' Comments: *For part (i), many candidates did not recognize that the expected emerging profit depends on the state at the beginning of the period. Most candidates did well on part (ii). Some candidates discounted the profit signature values at a rate other than the hurdle rate.*

Question 29

(a) The emerging profit in year 3, conditional on being in State *j* at time 2 is

$$Pr_3^{(j)} = \left({}_2V^{(j)} + P_3^{(j)} - E_3^{(j)}\right)(1 + i_t) - \sum_k E\ B_3^{jk} \ - \sum_k E_3 V^{jk}$$

where $i_t = 0.06$.

For State 0,

${}_2V^{(0)} = 12{,}000$

$P_3^{(0)} \ = \ 30{,}000$

$E_3^{(0)} = (0.05)P_3^{(0)} = 1500$

$E\ B_3^{01} = p_{x+2}^{01}\ \ 600{,}000 \ = \ (0.014)(600{,}000) = 8400$

$E\ B_3^{02} = p_{x+2}^{02}\ \ 1{,}000{,}000 \ = \ (0.014)(1{,}000{,}000) \ = \ 14{,}000$

$E\ B_3^{03} = p_{x+2}^{03}\ \ 1{,}000{,}000 \ = \ (0.004)(1{,}000{,}000) \ = \ 4000$

$E_3V^{00} = p_{x+2}^{00}\ {}_3V^{(0)} = (1 - 0.014 - 0.014 - 0.004)(9000) = 8712$

$E_3V^{01} = p_{x+2}^{01}\ {}_3V^{(1)} = \ (0.014)(210{,}000) \ = \ 2940$

So,

$Pr_3^{(0)} = (12{,}000 + 30{,}000 - 1500)(1.06) - 8400 - 14{,}000 - 4000 - 8712 - 2940$

$\qquad = 4878$

***Graders' Comments**: For those who did not achieve full marks, common errors included omitting the value of the benefit payable in case of death after CI, incorrectly including a value for the 500,000 benefit payable in case of transition from State 1 (at the beginning of the year) to State 2, or using the wrong expenses for year 3.*

(b) For State 1, we have

${}_2V^{(1)} = 280{,}000$

$P_3^{(1)} \ = \ 0$

$E_3^{(1)} = 100$

$E\ B_3^{12} = p_{x+2}^{12}\ \ 500{,}000 \ = \ (0.25)(500{,}000) = 125{,}000$

$E_3V^{11} = p_{x+2}^{11}\ {}_3V^{(1)} = \ (1 - 0.25)(210{,}000) \ = \ 157{,}500$

$Pr_3^{(1)} = (280{,}000 - 100)\ \ (1.06) \quad - 125{,}000 - 157{,}500 = \ 14{,}194$

***Graders' Comments**: Common errors included ignoring or using wrong expenses, and adding a premium when none is paid while in State 1.*

(c) $\Pi_3 \; = {}_2p_x^{00} \;\; Pr_3^{(0)} + {}_2p_x^{01} \;\; Pr_3^{(1)}$

$$
\begin{aligned}
{}_2p_x^{00} &= \; p_x^{00} \;\; p_{x+1}^{00} = (1 - p_x^{01} - p_x^{02} - p_x^{03})(\, 1 - p_{x+1}^{01} - p_{x+1}^{02} - p_{x+1}^{03}) \\
&= \; (1 - 0.01 - 0.008 - 0.004)(1 - 0.012 - 0.011 - 0.004) = 0.951594
\end{aligned}
$$

$${}_2p_x^{01} = \; p_x^{00} \;\; p_{x+1}^{01} + \; p_x^{01} \;\; p_{x+1}^{11} = \; (0.978)(0.012) + (0.01)((0.75) = 0.019236$$

$$\Pi_3 \; = (0.951594)(4878) + (0.019236)(14{,}194) = 4914.91$$

(d) $NPV(3) = \sum_{k=0}^{3} \Pi_k \;\; v_{10\%}^{k} \;\; = \; -500 - \frac{770}{1.1} + \frac{3536}{1.1^2} + \frac{4914.91}{1.1^3} = 5414.96$

Question 30

(a) $EPV(Premiums) = EPV(return\ of\ premiums) + EPV(payments) + EPV(expenses)$

$P \cdot (0.95\, \ddot{a}_{65:\overline{10}|} - 0.25\,)$

$$= (10\,P)\ {}_{10}E_{65}\, A^{\,1}_{75:\overline{10}|} + (36{,}000)\ {}_{20}E_{65}\, \ddot{a}_{85} + 900 + 100\ \ddot{a}_{65}$$

$$P = \frac{36{,}000(0.24381)(6.7993) + 900 + 100(13.5498)}{(0.95)(7.8435) - 0.25 - (10)(0.55305)(0.65142 - 0.44085)} = 10{,}259.385$$

(b) $Pr_0 = -E_0 - {}_0V = -7000 - 500 = -7500$

$Pr_t = {}_{t-1}V + CF_t - E_t + I_t - EDB_t - E\,{}_tV; \qquad t \geq 1$

where CF_t is the net cash flow received by the insurer at time t.

$$CF_t = \begin{cases} P = 10{,}259.385 & t = 0,1,\ldots,9 \\ 0 & t = 10,11,\ldots,19 \\ -36{,}000 & t = 20,21,\ldots \end{cases}$$

$Pr_1 = ({}_0V + P - (0.05P + 70))(1+i) - 0 - (1-0.03)(1 - 0.9\,q_{65})\ {}_1V$

$Pr_1 = (500 + (0.95)(10{,}259.385) - 70)(1.07) - (0.9648362)(10{,}150) = 1{,}095.68$

$Pr_{12} = ({}_{11}V - 70(1.02)^{11})(1+i) - 0.9q_{76}(10)(10{,}259.385) - (1-0)(1 - 0.9\,q_{76})\ {}_{12}V$

$Pr_{12} = (143{,}035 - 70(1.02)^{11})(1.07) - 0.0186012(102{,}593.85)$
$\qquad - (0.9813988)(151{,}210)$
$\quad = 2{,}648.64$

$Pr_{30} = ({}_{29}V - 36{,}000 - 70(1.02)^{29})(1+i) - 0 - (1-0)(1 - 0.9\,q_{94})\ {}_{30}V$

$Pr_{30} = (155{,}745 - 36{,}000 - 70(1.02)^{29})(1.07) - (0.8595532)(146{,}275) = 2{,}262.995$

(c)

Let NPV_1 be the EPV at the start of year 2 (time 1) of future emerging profits per policy in force.

$$NPV_1 = Pr_2\, v_{0.1}^1 + {}_1p_{x+1}^{00}\, Pr_3\, v_{0.1}^2 + {}_2p_{x+1}^{00}\, Pr_4\, v_{0.1}^3 + \ldots$$

$$NPV = \sum_{t=0}^{n} \pi_t\ v_{0.1}^t = 8860$$

where $\pi_0 = Pr_0$ and $\pi_t = {}_{t-1}p_x^{00}\, Pr_t,\ \ t \geq 1.$

$$\begin{aligned} NPV &= Pr_0 + Pr_1\, v_{0.1} + \left({}_1p_x^{00}\, Pr_2\, v_{0.1}^2 + {}_2p_x^{00}\, Pr_3\, v_{0.1}^3 + \ldots\right) \\ &= Pr_0 + Pr_1 v_{0.1} + (\text{EPV at time 0 of profits emerging in years 2, 3, …}) \\ &= Pr_0 + Pr_1\, v_{0.1} + {}_1p_x^{00}\, v_{0.1}\left(Pr_2\, v_{0.1} + {}_1p_{x+1}^{00}\, Pr_3\, v_{0.1}^2 + \ldots\right) \\ &= Pr_0 + Pr_1\, v_{0.1} + {}_1p_x^{00}\, v_{0.1} NPV_1 \end{aligned}$$

$$8860 = -7500 + \frac{1095.68}{1.1} + (1 - 0.03)(1 - 0.9q_{65})\ \frac{1}{1.1}\ \ NPV_1$$

$$NPV_1 = \frac{8860 + 7500 - 1095.68/1.1}{(0.9648362)/1.1} = \frac{15{,}363.9273}{0.877124} = 17{,}516.25$$

Graders' Comments: *This was a challenging question part. Only a minority of candidates achieved partial or full credit.*

Question 31

(a) EPV(payments + expenses)=100,000

$2000 + 500 + (B + 25)a_{\overline{65:\overline{10|}}} = 100{,}000$ where $a_{\overline{65:\overline{10|}}} = a_{\overline{10|}} + {}_{10}E_{65}\, a_{75}$

$$a_{\overline{65:\overline{10|}}} = \frac{1-v^{10}}{i} + (0.55305)(9.3178) = 7.72173 + 5.15321 = 12.87494$$

$$\Rightarrow \; B = \frac{100{,}000 - 2500 - (25)(12.87494)}{12.87494} = 7{,}547.848$$

(b) Note that at age 73, the annuity is within the 10-year guarantee period. So only states 0 and 1 are possible. Possible transitions are 0→0, 0→1, 1→1.

(i) $p_{73}^{01} = q_{73} = 0.014664$

(ii) $p_{73}^{02} = 0$
Since there is no possible transition from State 0 at 73 to State 2 at age 74since the annuity is within the guarantee period.

(iii) $p_{73}^{12} = 0$ Since payments would continue until age 76 if annuitant died before 73.

(c) At time 8, there are 2 years left in the guarantee period.

${}_8V^{(0)} = EPV$(future payments plus maintenance expenses)

$${}_8V^{(0)} = (B + 25)a_{\overline{65+8:\overline{10-8|}}}$$
$$= (7{,}547.848 + 25)a_{\overline{73:\overline{2|}}} \quad \text{where}$$
$$a_{\overline{73:\overline{2|}}} = a_{\overline{2|}} + {}_2E_{73}\, a_{75} = \frac{1-v^2}{i} + v^2\, {}_2p_{73}(9.3178)$$

$$= 1.85941 + (0.879036)(9.3178) = 10.050091$$

$$\Rightarrow \; {}_8V^{(0)} = 76{,}107.81$$

Graders' Comments: *Many students did not answer this part, but those that attempted it did well.*

(d)

In State 1, there are only 2 guaranteed payments to be made.

$$ {}_8V^{(1)} = (B+25)a_{\overline{2}|} $$
$$ = (7547.848 + 25)(1.85941) = 14{,}081.03 $$

Graders' Comments: *As with Part (c), many students did not answer this part. Those that did answer, did well.*

(e)

(i) $Pr_9^{(0)} = {}_8V^{(0)}(i_9 - i)$
$= (76{,}107.81)(0.06 - 0.05) = 761.08$

(ii) $Pr_9^{(1)} = {}_8V^{(1)}(i_9 - i)$
$= (14{,}081.03)(0.06 - 0.05) = 140.81$

(i) $\pi_9 = {}_8p_{65}Pr_9^{(0)} + (1 - {}_8p_{65})Pr_9^{(1)}$
$= (0.9295525)(761.08) + (0.0704475)(140.81)$
$= 717.38$

Graders' Comments: *Candidates found this part to be very challenging.*

Question 32

(a)

For T_{50}=11.8, we have K_{50}=11 which means that 12 premiums would be returned without interest.

Let S = 100,000 and P = 2000.

$$L^g = (S + 12P + 1000)v^{12} + 0.75\,P - 0.95\,P\,\ddot{a}_{\overline{12}|}$$

$$\ddot{a}_{\overline{12}|} = \frac{1 - 1.05^{-12}}{0.05/1.05} = 9.306414$$

$$L^g = (100{,}000 + 12(2000) + 1000)\,1.05^{-12} + 0.75(2000) - 0.95(2000)(9.306414)$$

$$L^g = 69{,}604.6773 + 1500 - 17{,}682.187 = 53{,}422.49$$

Graders' Comments: *Candidates generally did well in this part. Common mistakes were using 11.8 or 11 as the number of premiums or not properly including the return of premium benefit.*

(b) $$\mathrm{E}\left[L^g\right] = (S+1000)A_{50} + P\,{}_{10}E_{50}\left(10A_{60} + (IA)_{60}\right) + 0.05P\ddot{a}_{50} + 0.75P - P\ddot{a}_{50}$$
$$= -248.53$$

Graders' Comments: *Many candidates did not correctly allow for the return of premium benefit.*

(c)

$${}_{10}V^g = (S + 1000)A_{60} + 10P\,A_{60} + P(IA)_{60} - 0.95\,P\,\ddot{a}_{60}$$

$${}_{10}V^g = (100{,}000 + 1000)(0.029028) + 10(1000)(0.229028) + (2000)(6.63303) - 0.95(2000)(14.9041)$$

= 29,318.28 + 5805.60 + 13,266.06 – 28,317.79 = 20,072.15

Graders' Comments: *Candidates did better on this part than on Part (b) although the return of premium benefit was still a stumbling block for many.*

(d)

(i) Total gain: using the actual experience (*) during year 16,

$$1000({}_{15}V^g + P - e_{15}^*)(1 + i_{15}^*) - 1000\, q_{65}^*(S + 16P + E_{15}^*) \ - 1000\, p_{65}^*\, {}_{16}V^g$$

=1000(34,333.78+0.96(2000))(1.052)–7(100,000+32,000+2000) – 993(37,480.51)

= 38,138,976.56 – 938,000 – 37,218,146.43

= –17,169.87 i.e. a loss of 17,169.87

(ii) Gain by source:

<u>Gain from expenses (E)</u>:

Expected expenses – Actual expenses, valued at year end

$$[1000(0.05P)(1.05) + 1000\ \mathrm{q}_{65}(1000)] - [1000(0.04P)(1.05) + 1000\mathrm{q}_{65}(2000)]$$

= [105,000 + 1000(0.005915)(1000)] – [84,000 + 1000(0.005915)(2000)]

= 110,915 – 95,830 = **15,085** ➔ a gain from expenses of 15,085

<u>Gain from interest (I)</u>:

Actual interest earned – expected interest earned, using the actual expenses

$1000({}_{15}V^g + 0.96P)(i_{15}^* - 0.05)$= 1000 [34,333.78 + 0.96(2000)] (.002)

= **72,507.56**➔a gain from interest of 72,507.56

So the gain from interest is 87,553.78 – 15,046.22 = 72,507.56

<u>Gain from mortality (M)</u>:

Expected mortality cost – actual mortality cost, using the actual expenses and interest

$$[1000q_{65}(S + 16P + E_{15}^*) + 1000p_{65}\, {}_{16}V^g]$$
$$-[1000\ (0.007)(S + 16P + E_{15}^*) + 1000\ (0.993)\, {}_{16}V^g]$$

= 792,610 + 37,258,812.78 – 938,000 – 37,218,146.43 = **–104,723.65**

➔ a loss from mortality of 104,723.65

Graders' Comments: *Gain by interest was done the best while gain by mortality was done the worst. Some candidates answered part (d).ii before part (d).i and used the sum of the gains from part (d).ii to answer part (d).i, if done correctly, this received full credit. Some candidates forgot to multiply their gains by 1000 policies - a small deduction was made for this.*

(e)

The orders **MEI and EMI** had the same interest gain as EIM.

This can be justified a number of ways.

- The mortality cost is incurred at year end and has no impact of the interest earned during the year.
- Since actual premium expenses are different than expected, the gain from expenses (E) must be calculated before that from interest (I) to get the same value. So, E must precede I.
- Expenses affect the interest earned, so E must be calculated before I.

Graders' Comments: *Some candidates mistakenly commented on the total gain (i.e. stated that regardless of the order, the total gain will not change) - this did not receive any credit.*

Question 33

(a)

(i)

$$
\begin{aligned}
(IA)_x &= v\, q_x + 2v^2\, p_x\, q_{x+1} + 3v^3\, {}_2p_x\, q_{x+2} + 4v^4\, {}_3p_x\, q_{x+3} + \ldots \\
&= v\, q_x + v^2\, p_x\, q_{x+1} + v^3\, {}_2p_x\, q_{x+2} + v^4\, {}_3p_x\, q_{x+3} + \ldots \\
&\qquad + v^2\, p_x\, q_{x+1} + 2v^3\, {}_2p_x\, q_{x+2} + 3v^4\, {}_3p_x\, q_{x+3} + \ldots \\
&= A_x \\
&\qquad + v\, p_x \{v q_{x+1} + 2v^2\, p_{x+1}\, q_{x+2} + 3v^3\, {}_2p_{x+1}\, q_{x+3} + \ldots\} \\
&= A_x + v\, p_x\, (IA)_{x+1}
\end{aligned}
$$

(ii) $(IA)_{50} = A_{50} + v\, p_{50}\, (IA)_{51} = 5.8255$

$$
(IA)_{51} = \frac{5.8255 - A_{50}}{v\, p_{50}} = \frac{5.8255 - 0.18931}{0.998791/1.05} = 5.92516
$$

Graders' Comments: *Candidates did very well on this part. Where points were deducted, it was primarily from part (i) not being sufficiently rigorous.*

(b)

$$
\begin{aligned}
(200{,}000 + 1000)A_{50} + 5000((IA)_{50} - A_{50}) &= P((0.95)\, \ddot{a}_{50:\overline{20|}} - 0.1) \\
(195{,}000 + 1000)A_{50} + 5000(IA)_{50} &= P((0.95)\, \ddot{a}_{50:\overline{20|}} - 0.1)
\end{aligned}
$$

$$
37{,}104.76 + 29{,}127.5 = P((0.95)(12.8428) - (0.1))
$$

$$
P = \frac{66{,}232.26}{12.10066} = 5473.44
$$

Graders' Comments: *Candidates did very well on this part with most getting full credit.*

(c)

Recursively,

$$ {}_1V = \frac{(5473.44)(0.85)(1.05) - (0.001209)(201{,}000)}{0.998791} = 4647.6 $$

$$ {}_2V = \frac{[4647.6 + (5473.44)(0.95)](1.05) - (0.001331)(206{,}000)}{0.998669} = 10{,}078.9 $$

(d)

(i) Total gain: using the actual experience (*) during year 11,

$$ 1000({}_{10}V + P - e_{10}^{*})(1 + i_{10}^{*}) - 1000\, q_{60}^{*}(250{,}000 + E_{10}^{*}) - 1000\, p_{60\ 11}^{*}V $$

=1000(63,208 + 0.94(5473.44)) (1.06) – 6 (250,000+ 1100) – 994(71,217)

= 1000[72,454.2156 -1506.6 -70,789.698]

= 157,917.62   i.e. a gain per policy in force at 10 of 157.92

(ii) Gain by source:

<u>Gain from interest:</u>

Actual interest earned – expected interest earned,

$$ 1000(63{,}208 + 0.95(5473.44))\,(1.06) - [1000(63{,}208\ ) + 0.95(5473.44))\,(1.05)] $$

= 1000(68,407.768)(1.06−1.05)

= 684,077.68   i.e. a gain from interest of 684,077.68

<u>Gain from mortality:</u>

Expected mortality cost – actual mortality cost, using the actual interest

$$ [1000q_{60}(250{,}000 + E_{10}) + 1000p_{60\ 11}V^{g}] - [1000\, q_{60}^{*}(250{,}000 + E_{10}) - 1000\, p_{60\ 11}^{*}V^{g}\,] $$

= 852,898 + 70,975,004.6 – (1,506,000 + 70,789,698)

= - 467,795.37   i.e. a loss from mortality of 467,795.37

Gain from expenses:
Expected expenses – Actual expenses, valued at year end, using actual interest & mortality

$$(1000({}_{10}V^g + P - e_{10})(1.06) + 1000\, q^*_{60}(250{,}000 + E_{10}))$$
$$-[1000({}_{10}V^g + P - e^*_{10})(1.06) + 1000\, q^*_{60}(250{,}000 + E^*_{10})]$$

= 1000(0.05P-0.06P)(1.06) + 1000(0.006)(251,000 – 251,100)

= –58,618.46   i.e. a loss from expenses of 58,618.46

Graders' Comments: *A common error was not doing the calculations in the order given or not incorporating some of the actual experience.*

Question 34

(a) $Pr_0 = -3000 - (0.15)\,P = -3000 - (0.15)(80{,}000) = -15{,}000$

Graders' Comments: *Candidates generally did well here. Common mistakes included calculating a reserve when the question specifically stated V=0, or recalculating P when the question explicitly gave a gross premium.*

(b) $Pr_2 = (0.95P - 100)(1.07) - S\,q_{x+1}$

$= (0.95(80{,}000) - 100)(1.07) - 1{,}000{,}000\,(0.1)$

$= 81{,}213 - 100{,}000 = -18{,}787$

Graders' Comments*: Candidates generally did well here. Some candidates incorrectly applied survival probabilities that should have been applied in part (c).*

(c)

$$\Pi = (-15{,}000;\ 31{,}213;\ \ Pr_2\ {}_1p_x^{(\tau)})$$
$$= (-15{,}000;\ 31{,}213;\ -18{,}787\,(1-0.05)(.9))$$
$$= (-15{,}000;\ 31{,}213;\ -16{,}062.885)$$

Graders' Comments: *The most common mistake here was not treating the decrements as independent for the profit signature at time 2. Some candidates also forgot to include one of the two decrements entirely.*

(d)

NPV = $-15{,}000 + 31{,}213\, v_{0.2} - 16{,}062.885\, v_{0.2}^2$
$= -15{,}000 + 26{,}010.83 - 11{,}154.78 = -143.95$

(e)

Let j=IRR

$$-15{,}000 + 31{,}213\, v_j - 16{,}062.885\, v_j^2 = 0$$

⇨ $v_j = \dfrac{31{,}213 \pm \sqrt{31{,}213^2 - 4(15{,}000)(16{,}062.885)}}{2(16{,}062.885)} = \dfrac{31{,}213 \pm 3237}{32{,}125.77}$

⇨ $\boldsymbol{j} = -6.7\%\ or\ \mathbf{14.83}\%$ ➔ Keeping the positive rate, IRR= 14.83%.

(f)

The NPV is being calculated at a risk discount rate of 20% while the IRR uses a discount rate of 14.83. The 20% discount rate results in a negative because the discounted profits at the end of one year and at the end of two years when discounted at 20% are not sufficient to cover the pre-contract expenses. The NPV is positive for some rates of return less than 20%. The IRR>0 for discount rates less than IRR=14.83% (rates between -6.7% and 14.83%).

Graders' Comments: *Many candidates simply restated the question here and could not point to how it's possible to have a positive IRR but a negative NPV. Some mentioned the difference in discount rate vs IRR, but we were really looking for commentary of the timing of the cash flows.*

(g)

With a 15% lapse rate, ${}_1p_x^{(\tau)} = (1 - 0.05)(.85) = 0.8075$

The profit signature becomes

$\Pi = (-15{,}000;\ 31{,}213;\ Pr_2\ {}_1p_x^{(\tau)})$

$= (-15{,}000;\ 31{,}213;\ -18{,}787\ (1-0.05)(.85))$

$= (-15{,}000;\ 31{,}213;\ -15{,}170.50)$

And NPV at 20% is

NPV = $-15{,}000 + 31{,}213\ v_{0.2} - 15{,}170.50\ v_{0.2}^2$

= – 15,000 + 26,010.83 – 10,535.07 = 475.76

(h)

Lapse rates are much less predictable than mortality rates. A contract that is only profitable if people lapse is very vulnerable to the risk that policyholders lapse at much lower than the expected rates. The growth of the third party market for insurance makes lapsing much less attractive for policyholders, as a rational policyholder would not lapse a policy that could have positive value on the third party market.

Graders' Comments: *Very few candidates noted that the decision to lapse is the policyholder's, and there are many factors outside the control of the insurer to impact it.*

Question 35

(a)

- To set premiums
- To set reserves
- To measure profitability
- To stress test profitability
- To determine distributable surplus (for participating contracts)

Graders' Comments: *Most candidates that tried this part received full credit for it.*

(b)

$$
\begin{aligned}
{}_2V^n &= EPV\ of\ Future\ Benefits - EPV\ of\ Future\ Premiums \\
&= S\,v\,q_{x+2} - P = 1{,}000{,}000(1.05^{-1})(0.3) - 177{,}313.20 \\
&= 285{,}714.29 - 177{,}313{,}20 = 108{,}401.09
\end{aligned}
$$

Graders' Comments: *Most candidates received full credit for this part. Common errors involved an incorrect discount rate, including expenses, or including withdrawals.*

(c)

(i) $Pr_0 = -4000 - (0.15)G$
$= -4000 - (0.15)(210{,}000) = -35{,}500$

(ii) Lapse rate: w=0.1

$Pr_1 = (0.95G - E_0)(1 + i^*) - 1{,}000{,}000\, q_x - p_x\ (1 - w)\ {}_1V^n$
$= (0.95(210{,}000) - 100)(1.07) - 1{,}000{,}000(0.1) - (0.9)(0.9)(95{,}754)$
$= 213{,}358 - 100{,}000 - 77{,}560.74 = 35{,}797.26$

Graders' Comments: *Part (i) was done well by most candidates. In part (ii), many candidates failed to include withdrawals.*

(d)

(i) $\Pi_0 = Pr_0 = -35{,}500$
$\Pi_1 = Pr_1 = 35{,}797.26$
$\Pi_2 = p_x^{(\tau)}\, Pr_2 = (0.9)(0.9)(37{,}766) = 30{,}590.46$
$\Pi_3 = {}_2p_x^{(\tau)}\, Pr_3 = p_x^{(\tau)} p_{x+1}^{(\tau)}\, Pr_3 = (0.9)(0.9)(0.8)(0.9)(29{,}347) = 17{,}115.17$

(ii) $NPV = \sum_t \Pi_t \, v_{0.12}^t$

$$= -35{,}500 + \frac{35{,}797.26}{1.12} + \frac{30{,}590.46}{1.12^2} + \frac{17{,}115.17}{1.12^3} = 33{,}030.61$$

(iii) $PM = \frac{NPV}{G\left(1 + v_{0.12}^1 \, p_x^{(\tau)} + v_{0.12}^2 \, {}_2p_x^{(\tau)}\right)}$

$$= \frac{33{,}030.61}{210{,}000(1 + 0.7232143 + 0.4649235)} = \frac{33{,}030.61}{495{,}508.93}$$
$$= 0.07188 \quad or\ 7.19\%$$

Graders' Comments: *Many candidates failed to include withdrawals, and some candidates did not use the correct discount rates.*

(e) The NPV in terms of G is ($v = v_{0.12}$)

$$NPV = -4000 - (0.15)G$$
$$+\{(0 + \ 0.95G - 100)(1.07) - (1{,}000{,}000)(0.1) - (0.9)(0.9)(95{,}754)\}v$$
$$+\{(95{,}754 + 0.95G - 100)(1.07) - (1{,}000{,}000)(0.2)$$
$$- (0.8)(0.9)(108{,}401.09)\}\ (0.9)^2\ v^2$$
$$+\{(108{,}401.09 + 0.95G - 100)(1.07) - (1{,}000{,}000)(0.3)\}\ (0.9)^2\ (0.9)(0.8)\ v^3$$

$$NPV = G\left(-0.15 + 0.95(1.07)(v + (0.9)^2 v^2 + (0.9)^3(0.8)v^3)\right)$$
$$-4000 - 177{,}667.74\ v - 142{,}316.19\ v^2 - 107{,}377.52\ v^3$$
$$NPV = 1.8359304\ G - 352{,}514.70$$

$$PM = \frac{1.8359304\ G\ - 352{,}514.70}{2.18814\ G} = 0.1$$

⇨ $G = \frac{352{,}514.70}{1.8359304 - 0.218814} = \mathbf{217{,}989.69}$

Graders' Comments: *Very few candidates attempted this part of the problem. Candidates who attempted to write the NPV in terms of the new premium were rewarded significantly, even if the attempt was incomplete/incorrect.*

Question 36

(a) (i) $_1q_{[95]} = 1 - \frac{l_{96}}{l_{[95]}} = 0.45$

(ii) $_1q_{95} = 1 - \frac{l_{96}}{l_{95}} = 0.2143$

(iii) Typical select tables used for life insurance pricing assume lighter mortality during the select period, as the underwriting process selects lives that are healthier than average at inception. In this table, the mortality during the select period is heavier than ultimate mortality.

***Graders' Comments**: Candidates generally did well on part A. We awarded partial credit for candidates calculating the probability of survival, not death. On part iii we were looking for commentary on the q_xs themselves, and not necessarily the structure of the table (since select and ultimate tables come in many varieties).*

(b)

(i)

t	$_{t-1}V$	Prem	Annuity	Exp	Int	E_tV	Pr_t
0				300			–300
1	0	110,000	50,000	100	4,193	57,750	6,343
2	105,000	--	50,000	100	3,843	56,000	2,743

$$E_1V = p_{[95]} \times 105,000 = 57,750; \quad E_2V = p_{96} \times 80,000 = 56,000$$

(ii)

$$\Pi_0 = Pr_0 = -300; \quad \Pi_1 = Pr_1 = 6,343; \quad \Pi_2 = p_{[95]} Pr_2 = 1,509;$$
$$\Pi_3 = {}_2p_{[95]} Pr_3 = 130; \quad \Pi_4 = {}_3p_{[95]} Pr_4 = 82$$

$$\textit{where} \;\; p_{[95]} = \frac{550}{1000}, \; {}_2p_{[95]} = \frac{385}{1000}, \text{ and } {}_3p_{[95]} = \frac{195}{1000}$$

$$\left.\begin{aligned} NPV(1) &= NPV(0) + \Pi_1 \, v_{12\%} = 5,363; \\ NPV(2) &= NPV(1) + \Pi_2 \, v_{12\%}^2 = 6,566; \\ NPV(3) &= NPV(2) + \Pi_3 \, v_{12\%}^3 = 6,659; \\ NPV(4) &= NPV(3) + \Pi_4 \, v_{12\%}^4 = 6,711 \end{aligned}\right\}$$

(iii) The profit margin is 6,711/110,000=0.0610

Graders' Comments: *Candidates also generally did well here.*

(c) We now have

t	$_{t-1}V$	Prem	Annuity	Exp	Int	EtV	Pr_t
1	0	P	--	--	$0.07P$	30,195	1.07P-30,195
2	54,900	--	--	--	3,843	56,000	2,743

All the Π_t values are the same as in (a), except for the first year:

$$\Pi_1 = Pr_1 = 1.07P - 30{,}195$$

$$NPV = 0.061P = 6711 - 6343v_{12\%} + (1.07P - 30{,}195)v_{12\%}$$

$$\Rightarrow P = 28{,}973$$

Graders' Comments: *Many candidates did not attempt this question. For those who did, some incorrectly attempted to use recursion to calculate a new premium based on then new reserve, which doesn't work in part because of the constraint to maintain the same profit margin.*

(d) Disadvantages:

Under the deferred annuity, the policyholder would pay 78,973 (annuity plus premium) in the first year, 50,000 in the second year, conditional on survival, and nothing thereafter. The total cost for a survivor, is 128,973. Under the annuity-due, the total cost is 110,000. So, for a 1-year survivor, even allowing for interest, the deferred annuity is more expensive.

If death occurs in the first two years, the annuitant and/or beneficiaries will receive nothing, losing the single premium.

Advantages:

An advantage of the deferred annuity is that it is less costly for lives who die in the first year, whose outlay is 78,973 compared with 110,000 for the annuity-due, thereby leaving more available for bequest under the SPDA, in the event of early death.

Question 37

(a) Reasons include:

- To compete for new employees
- To retain employees in productive years
- To facilitate turnover of employees at older ages
- To offer tax efficient remuneration
- As a tool in negotiations with unions (or other employee collective bargaining units)
- To fulfill responsibility to provide for long-serving employees.
- To improve morale of employees

(b) Let S_x denote the salary earned in year of age x to $x+1$. We have

$S_x = (50,000)(1.03)^{x-38}$.

The EPV of the death benefit is:

$(2S_{62})q_{62}(1.04)^{-1} + (2S_{63})_{1|}q_{62}(1.04)^{-2} + (2S_{64})_{2|}q_{62}(1.04)^{-3}$

Where $q_{62} = 0.08$; ${}_{1|}q_{62} = (0.92)(0.09) = 0.0828$; ${}_{2|}q_{62} = (0.92)(0.91)(0.10) = 0.08372$

So the EPV of the death benefit is

$$(2)(50,000)(1.03)^{24}[0.08(1.04)^{-1} + (1.03)(0.0828)(1.04)^{-2} + (1.03)^2(0.08372)(1.04)^{-3}]$$
$$= 47,716$$

(c) The EPV of the retirement benefit is

$$(0.03)(27)(FAS)({}_3p_{62})(1.04)^{-3}\ddot{a}_{65}$$

Where FAS is the final average salary =

$$FAS = 50,000\left(\frac{(1.03)^{24} + (1.03)^{25} + (1.03)^{26}}{3}\right) = 104,719$$

So EPV is

$$(0.03)(27)(104,719)(0.75348)(0.88900)(4.7491) = 269,833$$

Graders' Comments: *Parts (b) and (c) were answered well. The most common minor error was counting years of service incorrectly, which resulted in a small penalty for part (b), and none for (c) if the answer was consistent with (b).*

Question 38

(a) Let S_x denote the salary in year of age x to $x+1$. The projected salaries in the final three years are:

$$S_{62} = (80,000)(1.04)^{17} = 155,832$$
$$S_{63} = (80,000)(1.04)^{18} = 162,065$$
$$S_{64} = (80,000)(1.04)^{19} = 168,548$$
$$FAS = \frac{155,832 + 162,065 + 168,548}{3} = 162,148$$

So the projected benefit per month is
25[(0.02)(100,000)+(0.03)(62,148)]/12=8050.92

(b) The Replacement Ratio (RR) is $\dfrac{(12)(8051)}{168,548} = 57.3\%$

Graders' Comments: *This part was done well. The most common error was using the final average salary instead of the final year's salary in the denominator.*

(c) The EPV is

$$EPV = (8051)(12)\ddot{a}_{65}^{(12)}$$

$$\ddot{a}_{65}^{(12)} = \frac{1 - A_{65}^{(12)}}{d^{(12)}} = \frac{1 - 0.470}{0.04869} = 10.885$$

$$\Rightarrow EPV = 1,051,620$$

Graders' Comments: *Many candidates gained full credit on this part. The most common error was forgetting to multiply the monthly benefit by 12*

(d) We require (0.8-0.573) = 0.227 replacement ratio for the DC plan.
The cost of adding 22.7% RR is

$$(0.227)(168,548)\ddot{a}_{65}^{(12)} = 416,464$$

The accumulated contributions to age 65, where c is the contribution rate, are

$$cS_{45}\left((1.07)^{20}+(1.04)(1.07)^{19}+\ldots+(1.04)^{19}(1.07)\right)$$
$$=cS_{45}\left(\frac{(1.07)^{20}-(1.04)^{20}}{1-(1.04)/(1.07)}\right)=(4,789,500)c$$

Equating the contributions and the benefit value gives

$$c=8.70\%$$

Graders' Comments: *This part was less well done, although a number of strong candidates scored full marks. The most common errors, which earned partial credit, were miscalculating the geometric series, or omitting either the 4% or 7% terms in the series.*

Question 39

(a) *RR* denotes the replacement ratio, *FAS* denotes the final average salary, S_x denotes salary in the year of age x to $x+1$. Then

$$RR = \frac{(9.5)(900) + (15.5)(FAS)(0.03)}{S_{64}}$$

$$S_{64} = (30,000)(1.02)^{24} = 48,253$$

$$FAS = (30,000)\left(\frac{1.02^{22} + 1.02^{23} + 1.02^{24}}{3}\right) = 47,313$$

$$\Rightarrow RR = \frac{30,551}{48,253} = 63.3\%$$

(b)

$$RR = \frac{900n + (25-n)(47,313)(0.03)}{48,253} \geq 0.65$$

$$\Rightarrow \frac{35,485 - 519.4n}{48,253} \geq 0.65 \Rightarrow n \leq 7.9 \text{ years}$$

(c) The total accrued benefit based on the first 15 years of employment is

$$(7)(900) + (8)(0.03)\left(\frac{S_{52} + S_{53} + S_{54}}{3}\right) = 15,615$$

The required benefit is $(0.65)(\mathrm{S}_{64}) = 31,364$

So the annuity payments are 15,749 per year, requiring premium of P where

$$P\ddot{a}_{55:\overline{10}|} = ({}_{10}E_{55})(15,749)(\ddot{a}_{65})$$

$$\Rightarrow P(8.0192) = (0.59342)(15,749)(13.5498)$$

$$\Rightarrow P = 15,791$$

Question 40

(a) The individual has a Defined Benefit plan. A Defined Contribution pension plan specifies how much an employer will contribute, usually as a percentage of salary, into a plan. The plan may allow employees to also contribute. The contributions are accumulated in a notional account which is available to the employee when they leave the company. The contributions may be set to meet a target benefit level, but the actual retirement income may be well below or above the target, depending on investment experience.

(b)

To complete Parts (b) and (c), we note that the individual will retire at age 60 or 61. Therefore we will need to know how much benefit has been accrued for both 60 and 61. We will also need to know the monthly annuity values at age 60 and 61. Using the 2-term Woolhouse approximation we have

$$\ddot{a}_{60}^{(12)} = \ddot{a}_{60} - \frac{11}{24} = 14.4458 \quad \text{and} \quad \ddot{a}_{61}^{(12)} = \ddot{a}_{61} - \frac{11}{24} = 14.1908$$

(i)

Under the Projected Unit Credit cost method, the actuarial accrued liability is the actuarial present value of the projected benefit. The projected benefit is equal to the final average salary at the decrement date multiplied by service as of the valuation date and by the accrual rate.

We have the following information.

Projected Final Average Salary at 60	$50,000[(1.03)^3 + (1.03)^4 + (1.03)^5]/3 = 56,292$
Projected Final Average Salary at 61	$50,000[(1.03)^4 + (1.03)^5 + (1.03)^6]/3$ $= (56,292)(1.03) = 57,981$
Service at valuation date	25
Accrual Rate	0.016
Projected Benefit for retirement at 60 (PB_{60})	(56,292)(0.016)(25) = 22,517
Projected Benefit for retirement at *61* (PB_{61})	(57,981)(0.016)(25) = 23,192

The actuarial accrued liability is the actuarial present value (as of the valuation date) of the projected benefit and is given by

$$\text{AAL}_{55} = PB_{60} \cdot \ddot{a}_{60}^{(12)} \cdot \frac{r_{60}^{exact}}{l_{55}^{(\tau)}} \cdot v^5 + PB_{61} \cdot \ddot{a}_{61}^{(12)} \cdot \frac{r_{60}^{During} + i_{60} + l_{61}^{(\tau)}}{l_{55}^{(\tau)}} v^6$$

$$= (22,517)(14.4458)\left(\frac{27,925.6}{104,687.7}\right)(1.05)^{-5} + (23,192)(14.1908)\left(\frac{6187.6 + 61.9 + 58,699.9}{104,687.7}\right)(1.05)^{-6}$$

$$= 220,351$$

(ii)
We now need the accrued benefit at age 56

Projected Benefit for retirement at 60 (PB_{60})	(56,292)(0.016)(26) = 23,417
Projected Benefit for retirement at 61 (PB_{61})	(57,981)(0.016)(26) = 24,120

The actuarial accrued liability at December 31, 2016 is given by

$$\text{AAL}_{56} = PB_{60} \cdot \ddot{a}_{60}^{(12)} \cdot \frac{r_{60}^{exact}}{l_{56}^{(\tau)}} \cdot v^4 + PB_{61} \cdot \ddot{a}_{61}^{(12)} \cdot \frac{r_{60}^{During} + i_{60} + l_{61}^{(\tau)}}{l_{56}^{(\tau)}} v^5$$

$$= (23,417)(14.4458)\left(\frac{27,925.6}{102,307.9}\right)(1.05)^{-4} + (24,120)(14.1908)\left(\frac{6187.6 + 61.9 + 58,699.9}{102,307.9}\right)(1.05)^{-5}$$

$$= 246,221$$

Then

${}_tV + C_t = EPV$ of benefits for mid-year exits $+ v \cdot {}_1p_x^{(\tau)} \cdot {}_{t+1}V$ where:

$C_t =$ Normal Cost for year t to $t+1$ and ${}_tV$ is the Actuarial Accrued Liability at time t

Note that EPV of benefits for mid-year exits is zero. Then:

${}_tV + C_t = EPV$ of benefits for mid-year exits $+ v \cdot {}_1p_x^{(\tau)} \cdot {}_{t+1}V$

$$220,351 + C_t = 0 + (1.05)^{-1}\left(\frac{102,307.9}{104,687.7}\right)(246,221)$$

$$C_t = 8815$$

(c) (i)

Under the Traditional Unit Credit cost method the actuarial accrued liability (AAL) is the actuarial present value of the accrued benefit on the valuation date.

The formula for the accrued benefit, B, is

$$B_{55} = 0.016 \cdot 25 \cdot 50,000 \cdot \left(\frac{1 + (1.03)^{-1} + (1.03)^{-2}}{3} \right) = 19,423$$

and thus

$$\begin{aligned} \text{AAL}_{55} &= B_{55} \cdot \ddot{a}_{60}^{(12)} \cdot \frac{r_{60}^{exact}}{l_{55}^{(\tau)}} \cdot v^5 + B_{55} \cdot \ddot{a}_{61}^{(12)} \cdot \frac{r_{60}^{during} + i_{60} + l_{61}^{(\tau)}}{l_{55}^{(\tau)}} v^6 \\ &= B_{55} \left(\ddot{a}_{60}^{(12)} \cdot \frac{r_{60}^{exact}}{l_{55}^{(\tau)}} \cdot v^5 + \ddot{a}_{61}^{(12)} \cdot \frac{r_{60}^{during} + i_{60} + l_{61}^{(\tau)}}{l_{55}^{(\tau)}} v^6 \right) \end{aligned}$$

$$= 19,423 \left[(14.4458) \left(\frac{27,925.6}{104,687.7} \right) (1.05)^{-5} + (14.1908) \left(\frac{6,187.6 + 61.9 + 58,699.9}{104,687.7} \right) (1.05)^{-6} \right]$$

$$= 186,248$$

(ii)

For the NC, we must calculate the expected accrued benefit, B_{61}, one year after the valuation date.

$$B_{56} = 0.016 \cdot 26 \cdot 50,000 \cdot \left(\frac{1.03 + 1 + 1.03^{-1}}{3} \right) = 20,806$$

$$\text{AAL}_{56} = B_{56} \cdot \left(\ddot{a}_{60}^{(12)} \cdot \frac{r_{60}^{exact}}{l_{56}^{(\tau)}} \cdot v^4 + \ddot{a}_{61}^{(12)} \cdot \frac{r_{60}^{during} + i_{60} + l_{61}^{(\tau)}}{l_{56}^{(\tau)}} v^5 \right)$$

$$= 20,806 \left[(14.4458) \left(\frac{27,925.6}{102,307.9} \right) (1.05)^{-4} + (14.1908) \left(\frac{6,187.6 + 61.9 + 58,699.9}{102,307.9} \right) (1.05)^{-5} \right]$$

$$= 214,358$$

Then:

${}_tV + C_t = EPV$ of benefits for mid-year exits $+ v \cdot {}_1 p_x^{(\tau)} \cdot {}_{t+1}V$ where:

C_t = Normal Cost for year t to $t+1$ and ${}_tV$ is the Actuarial Accrued Liability at time t

Note that EPV of benefits for mid-year exits is zero. Then:

$${}_tV + C_t = EPV \text{ of benefits for mid-year exits} + v \cdot {}_1 p_x^{(\tau)} \cdot {}_{t+1}V$$

$$186,248 + C_t = 0 + (1.05)^{-1}\left(\frac{102,307.9}{104,687.7}\right)(214,358)$$

$$C_t = 13,262$$

(d)

Under both funding approaches, the contribution rate tends to increase as the member acquires more service and gets closer to retirement. The TUC contributions start smaller than the PUC contributions and, rise more steeply, ending at considerably more than the PUC contribution. This is true because the TUC contributions do not project future salary increases or future service credit while PUC contributions are based on salary with projected future salary increases. Therefore, the TUC contributions in any given year must reflect the full increase in the salary and the additional year of service now reflected in the accrued benefit at the end of the year of service which were not reflected in the accrued benefit at the end of the year of service. The PUC contribution will only reflect the additional year of service; it does not need to reflect the salary increase as this is already allowed for in the accrued liability at the beginning of the year.

Question 41

(a)

$$S_{64} = 40{,}000\left(1+\frac{0.036}{12}\right)^{(64-30)(12)} s^{(12)}_{\overline{1|}\,i=1.003^{12}-1} = 138{,}044.47$$

The monthly pension, B, is:
$B = (0.02)(35)S_{64}/12 = 8052.594$

Graders' Comments: *Many candidates did well on this part, especially those who drew a timeline to show the increasing monthly earnings. For those who did not receive full credit, the most common error was to calculate the final year salary, S_{64}, incorrectly, for which a small deduction was applied.*

(b)

$$EPV = (0.02)(35)S_{64}\,\ddot{a}^{(12)}_{\overline{65:\overline{10|}}} = (0.02)(35)(138{,}044.47)(13.38208) = 1{,}293{,}125.93$$

where

$$\ddot{a}^{(12)}_{\overline{65:\overline{10|}}} = \ddot{a}^{(12)}_{\overline{10|}} + {}_{10}E_{65}\,\ddot{a}^{(12)}_{75} = \ddot{a}^{(12)}_{\overline{10|}} + {}_{10}E_{65}\,(\ddot{a}_{75} - 11/24)$$

$$= \left(\frac{1-1.05^{-10}}{12(1-1.05^{-1/12})}\right) + (0.55305)(10.3178 - 11/24) = 13.38208$$

Graders' Comments: *Common errors included calculating the EPV at the wrong date, using annual payments, incorrectly calculating the EPV of the 10-year guaranteed annuity and using UDD instead of the Woolhouse formula.*

(c)

The accumulated value of contributions, AV, is

$$AV = (0.06)\left(\frac{40{,}000}{12}\right) \text{ x } [1.008^{419} + (1.003)(1.008^{418}) + \cdots + (1.003^{419})(1.008^{0})]$$

$$= 200\left[(1.008^{419})\left(\frac{1-\left(\frac{1.003}{1.008}\right)^{420}}{1-\left(\frac{1.003}{1.008}\right)}\right)\right] = (200)(28.181425)(176.627679)$$

$$= 995{,}523.94$$

Graders' Comments: *Most candidates found this challenging. Those who used a timeline to show the contribution amounts found it helpful in answering this part.*

(d)

Since the lump sum of 1,293,126 is larger than the value of contributions at 9.6% (995,524), the IRR that earned by taking the lump sum is more than 9.6% convertible monthly.

Graders' Comments: *Many candidates tried to answer the question in general terms instead of comparing the options using the results from parts b) and c).*

(e)

The revised monthly benefit, B^*, is such that

$$EPV = 1{,}293{,}125.93 = (12\,B^*)\,\ddot{a}_{65}^{(12)} = (12\,B^*)(\ddot{a}_{65} - 11/24)$$

$$B^* = \frac{1{,}293{,}125.93}{12\left(13.5498 - \frac{11}{24}\right)} = 8{,}231.35$$

Graders' Comments: *The candidates who attempted this part did well. Some candidates used the AV of contributions instead of the EPV of benefits, for which no credit was given.*

(f)

(i) Adverse selection in insurance refers to a situation where a policyholder makes a decision based on asymmetric information about the risk, i.e. information known to the policyholder but unknown to the insurer. Policyholders who represent higher risk tend to buy more insurance.
Here, adverse selection would occur whenever an employee would choose the most advantageous option at the time of retirement based on his/her health and other risk factors unknown to the pension plan sponsor.

(ii) Adverse selection is likely to <u>increase the cost</u> to the plan. The healthiest employees mostly choosing the life annuity and those in poor health choosing the lump sum.

Graders' Comments: *Well-prepared candidates provided a coherent rationale as to why the costs to the plan would increase due to adverse selection. The candidates who discussed the pricing issues rather than the impact on the plan's costs received no credit for part (ii).*

Question 42

(a) $_0V = AL_0 = (TPE)_{63} \cdot \alpha \cdot \left(\frac{r_{63}}{l_{63}} v^{0.5}\, \ddot{a}_{63.5}^{(12)} + \frac{r_{64}}{l_{63}} v^{1.5}\, \ddot{a}_{64.5}^{(12)} + \frac{r_{65}}{l_{63}} v^{2}\, \ddot{a}_{65}^{(12)}\right)$

$$AL_0 = (2{,}500{,}000)(0.02)\left(\frac{4515.2}{47{,}579.3}(1.05)^{-0.5}(13.514) + \frac{4061.0}{47{,}579.3}(1.05)^{-1.5}(13.231) + \frac{38{,}488.3}{47{,}579.3}(1.05)^{-2}(13.086)\right)$$

$$= (50{,}000)(1.251550 + 1.049600 + 9.601498) = 595{,}132.40$$

Graders' Comments: *Performance on this part was mixed. Candidates either did very well, earning full credit, or not at all well. The most common error was to incorrectly value the liability for mid-year exits.*

(b) $AL_0 + NC = EPV(\textit{benefits for mid-year exits}) + v\, p_{63}^{00}\, AL_1$

where

$$EPV(\textit{benefits for mid-year exits}) = ((TPE)_{63} + 0.5\, s_{63})\, \alpha\, \frac{r_{63}}{l_{63}} v^{0.5}\, \ddot{a}_{63.5}^{(12)}$$
$$= \big(2{,}500{,}000 + (0.5)(160{,}000)\big)(0.02)(1.25155)$$
$$= 64{,}579.98$$

$$AL_1 = ((TPE)_{63} + s_{63}) \cdot \alpha \cdot \left(\frac{r_{64}}{l_{64}} v^{0.5}\, \ddot{a}_{64.5}^{(12)} + \frac{r_{65}}{l_{64}} v^{1}\, \ddot{a}_{65}^{(12)}\right)$$

$$v\, p_{63}^{00}\, AL_1 = ((TPE)_{63} + s_{63}) \cdot \alpha \cdot \left(\frac{r_{64}}{l_{63}} v^{1.5}\, \ddot{a}_{64.5}^{(12)} + \frac{r_{65}}{l_{63}} v^{2}\, \ddot{a}_{65}^{(12)}\right)$$
$$= (2{,}500{,}000 + 160{,}000)(0.02)(1.0496 + 9.601498) = 566{,}638.41$$

$$NC = 64{,}579.98 + 566{,}638.41 - 595{,}132.40 = 36{,}085.99$$

Alternatively,

$$NC = s_{63} \cdot \alpha \cdot \left[(0.5)\, \frac{r_{63}}{l_{63}} v^{0.5} \ddot{a}_{63.5}^{(12)} + \frac{r_{64}}{l_{63}} v^{1.5} \ddot{a}_{64.5}^{(12)} + \frac{r_{65}}{l_{63}} v^{2} \ddot{a}_{65}^{(12)}\right]$$

$$= (160{,}000)(0.02)[(0.5)(1.25155) + 1.0496 + 9.601498] = 36{,}085.99$$

Graders' Comments: *Partial credit was awarded to candidates who showed some understanding of the question by writing down some relevant formulas before evaluating them numerically.*

Question 43

(a)

(i) Final Average Salary (FAS) between ages 64 and 65 is projected to be

$$S_{39}(1.03)^{54-39}(1.025)^{64-54} = 110{,}000(1.557967)(1.2800845)$$
$$= 219{,}376.23$$

$$AL = 219{,}376.23(0.02)(20)\,{}_{25}E_{40}\ddot{a}_{65}^{(12)} = (87{,}750.49)\,{}_{25}E_{40}\ddot{a}_{65}^{(12)}$$

where

$${}_{25}E_{40} = v^{25}\ {}_{25}p_{40} = {}_{20}E_{40}\ {}_{5}E_{60} = (0.36663)(0.76687) = 0.281158$$

$$\ddot{a}_{65}^{(12)} = \ddot{a}_{65} - \frac{11}{24} = 13.5498 - \frac{11}{24} = 13.09147$$

$$\Rightarrow\ AL = (87{,}750.49)(0.281158)(13.09147) = 322{,}989.50$$

$AL_0 = 322{,}989.50$ and $AL_1 = 219{,}376.23(0.02)(21)\,{}_{24}E_{41}\ddot{a}_{65}^{(12)}$

$$NC = v\,p_{40}\,AL_1 - AL_0 = \frac{AL_0}{20} = 16{,}149.48;$$

$$NC\ rate = \frac{NC}{S_{40}} = \frac{16{,}149.48}{110{,}000(1.03)} = 0.142537$$

Graders' Comments: *Candidates did very well on both parts, up to the normal cost. Many candidates made errors concerting the normal cost to a normal cost rate.*

(b)

(i) Per unit of FAS, the EPV of the retirement benefits are

- <u>With partner at retirement</u>, $EPV^{p} = \ddot{a}_{65}^{(12)} + 0.5\,\ddot{a}_{65|65}^{(12)}$
$$= 13.09147 + 0.93335 = 14.02482$$

where

$$\ddot{a}_{65|65}^{(12)} = \ddot{a}_{65}^{(12)} - \ddot{a}_{65:65}^{(12)}$$
$$= \left(\ddot{a}_{65} - \frac{11}{24}\right) - \left(\ddot{a}_{65:65} - \frac{11}{24}\right)$$
$$= \left(13.5498 - \frac{11}{24}\right) - \left(11.6831 - \frac{11}{24}\right) = 1.8667$$

- Single, $EPV^s = 1.05\,\ddot{a}_{65}^{(12)} = 13.74604$

$$\begin{aligned} AL &= 219{,}376.23(0.02)(20)\,{}_{25}E_{40}\big((0.9)EPV^p + (0.1)EPV^s\big) \\ &= 219{,}376.23(0.02)(20)(0.281158)\big((0.9)(14.02482) + (0.1)(13.74604)\big) \\ &= 219{,}376.23(0.02)(20)(0.281158)(13.99694) \\ &= 345{,}329.04 \end{aligned}$$

(ii) The revised NC rate is

$NC = AL/20 = 345{,}329.04/20 = 17{,}266.45$

$$NC\ rate = \frac{NC}{S_{40}} = \frac{17{,}266.45}{110{,}000(1.03)} = 0.1524$$

***Graders' Comments**: This part was challenging. It required candidates to use the information in a new context. Many candidates did not use the 90% married at retirement information. A number discounted the benefit using ${}_{25}E_{40:40}$ instead of ${}_{25}E_{40}$, as if the member and partner had to survive to age 65 to receive any retirement benefit.*

(c)

The EPV of the reversionary partner benefits would decrease.

Sample explanation:

Since $\ddot{a}_{x|y} = \ddot{a}_y - \ddot{a}_{x:y}$ where $\ddot{a}_y$ does not change, the change in $\ddot{a}_{x|y}$ would be the opposite of the change in $\ddot{a}_{x:y}$.

Fixing the time of death of (x) and considering the two cases:

- (x) dies before (y): positive correlation means (y) will now tend to die younger than under the independence case but since payments stop at the death of (x) => no change
- (y) dies before (x): positive correlation means (y) will now tend to die older (closer to (x)). Here payments stop at the death of (y) => EPV increases on average.

Conclusion: $\ddot{a}_{x|y}$ decreases.

***Graders' Comments:** Many candidates used the broken heart syndrome as an example of positively correlated lifetimes and argued that the value of the reversionary annuity would decrease. This is answer (if complete) was awarded full credit. A few presented a more general argument to justify the decrease in EPV when lifetimes are positively correlated.*

Question 44

(a)

$$AL = 20 \times 8 \times 45{,}000 \times 0.02 \times {}_{30}E_{35} \times \ddot{a}_{65}^{(12)}$$
$$+5 \times 25 \times 62{,}000 \times 0.02 \times {}_{5}E_{60} \times \ddot{a}_{65}^{(12)}$$
$$+32{,}000 \times \ddot{a}_{70}^{(12)}$$
$$\ddot{a}_{65}^{(12)} = \ddot{a}_{65} - \frac{11}{24} = 13.0915; \quad \ddot{a}_{70}^{(12)} = 11.5500; \quad {}_{30}E_{35} = 0.21981$$
$$\Rightarrow AL = 20 \text{ x } 20{,}719 + 5 \text{ x } 311{,}223 + 369{,}599$$
$$= 414{,}380 + 1{,}556{,}115 + 369{,}599 = 2{,}340{,}094$$

(b)

NC for age 35 group:

$$414{,}380 \times \left(1.025 \times \frac{9}{8} - 1\right) = 63{,}452$$

NC for age 60 group:

$$1{,}556{,}111 \times \left(1.025 \times \frac{26}{25} - 1\right) = 102{,}704$$

Total Salary

$$45{,}000 \times 20 \times 1.025 + 62{,}000 \times 5 \times 1.025 = 922{,}500 + 317{,}750 = 1{,}240{,}250$$

Normal Contribution Rate

$$\frac{63{,}452 + 102{,}704}{1{,}240{,}250} = 13.397\%$$

(c) Using the numbers calculated in part (b), we have

$$\text{NC} = \frac{102{,}704}{317{,}750} = 32.32\%$$

(d) Under PUC the curve of contribution rates by age is less steep, so the NC at age 60 will be closer to the NC at age 35, implying that the change will be smaller. The reason is that the PUC pre-pays for future pay increases on accrued benefits, while the TUC does not. That means that TUC NC rates at older ages must pay for the new accrued benefit, and in addition must pay to upgrade all past accruals for the most recent pay rise. This creates a very steep curve of contribution rates at older ages.

Comment: Candidates who used a sketch of the PUC and TUC conrribution rates to support their justification were generally given full credit (assuming the sketch was correct).

Question 45

(a)

i) $$AL_0 = \alpha\ (TPS)_{63}\ x\ \left\{\frac{r_{63}}{l_{63}}\ v^{0.5}\ \ddot{a}^{(12)}_{63.5}\ + \frac{r_{64}}{l_{63}}\ v^{1.5}\ \ddot{a}^{(12)}_{64.5}\ + \frac{r_{65}}{l_{63}}\ v^{2}\ \ddot{a}^{(12)}_{65}\ \right\}$$

$$= (0.02)\ (2{,}400{,}000)\ \cdot\ \left\{\frac{4515}{47{,}579}\ 1.05^{-0.5}(13.5139) + \right.$$

$$\left. + \frac{4061}{47{,}579}\ 1.05^{-1.5}\ (13.2312)\ + \frac{38{,}488}{47{,}579}\ 1.05^{-2}\ (13.087)\right\}$$

$$=\ 48{,}000\ \{1.2515 + 1.0496 + 9.6022\}\ =\ 571{,}358.40$$

ii) $$NC = \alpha\ S_{63}\ \cdot\ \left\{0.5\ \ \frac{r_{63}}{l_{63}}\ v^{0.5}\ \ddot{a}^{(12)}_{63.5}\ + \frac{r_{64}}{l_{63}}\ v^{1.5}\ \ddot{a}^{(12)}_{64.5}\ + \frac{r_{65}}{l_{63}}\ v^{2}\ \ddot{a}^{(12)}_{65}\ \right\}$$

$$= (0.02)\ (170{,}000)\ \left\{\begin{matrix} 0.5\ \frac{4515}{47{,}579}\ 1.05^{-0.5}(13.5139) + \frac{4061}{47{,}579}\ 1.05^{-1.5}\ (13.2312) \\ + \frac{38{,}488}{47{,}579}\ 1.05^{-2}\ (13.087) \end{matrix}\right\}$$

$$=\ 3{,}400\ \left\{\frac{1.2515}{2} + 1.0496 + 9.6022\right\}\ =\ 38{,}344$$

Alternatively,

$$NC = vp_{63}AL_1 - AL_0 + EPV(\text{mid-year retirements})$$

$$= \left(\frac{1}{1.05}\right)\left(\frac{42{,}805}{47{,}579}\right)(638{,}995.53) - 571{,}358.40 + 62{,}199.55 = 38{,}346$$

where

$$AL_1 = \alpha\ [(TPS)_{63} + S_{63}]\ \cdot \left\{\frac{r_{64}}{l_{64}}\ v^{0.5}\ \ddot{a}^{(12)}_{64.5}\ + \frac{r_{65}}{l_{64}}\ v^{1}\ \ddot{a}^{(12)}_{65}\ \right\}$$

$$=\ (0.02)(2{,}570{,}000)\ \ \left\{\frac{4061}{42{,}805}\ \left(\frac{13.2312}{1.05^{0.5}}\right) + \frac{38{,}488}{42{,}805}\ \left(\frac{13.087}{1.05}\right)\right\}$$

$$= 51{,}400\ \ \{1.22502 + 11.2068\} = 638{,}995.53$$

$$EPV(\text{mid-year retirements}) = \alpha\ [(TPS)_{63} + 0.5\ S_{63}] \cdot \left\{\frac{r_{63}}{l_{63}}\ v^{0.5}\ \ddot{a}^{(12)}_{63.5}\ \right\}$$

$$= (0.02)\ [2{,}485{,}000]\ \{1.2515\} = 62{,}199.55$$

(b)

i) $AL_0 = 4 \;\; \alpha\,(TPS)_{63}\, x \left\{\frac{d_{63}}{l_{63}}\, v^{0.5} \;\; + \frac{d_{64}}{l_{63}}\, v^{1.5} \;\right\}$

$$= 4(0.02)\,(2{,}400{,}000)\, x \left\{\frac{213.9}{47{,}579}\, 1.05^{-0.5} \;+ \frac{215.1}{47{,}579}\, 1.05^{-1.5}\right\}$$

$$= \; 192{,}000\,\{0.004387 + 0.004202\} \;=\; 1649.09$$

ii) $NC = 4\alpha\, S_{63}\, x \left\{0.5 \;\; \frac{d_{63}}{l_{63}}\, v^{0.5} \;\; + \frac{d_{64}}{l_{63}}\, v^{1.5}\right\}$

$$= 4(0.02)\,(170{,}000)\, x \left\{0.5\, \frac{213.9}{47{,}579}\, 1.05^{-0.5} + \frac{215.1}{47{,}579}\, 1.05^{-1.5}\right\}$$

$$= \; 13{,}600\,\{\frac{0.004387}{2} + 0.004202\} \;=\; 86.98$$

Alternatively,
$NC = vp_{63}AL_1 - AL_0 + EPV$(mid-year deaths)

$$= \left(\frac{1}{1.05}\right)\left(\frac{42{,}805}{47{,}579}\right)(1008.26) - 1649.09 + 872.20 = 86.95$$

where

$AL_1 = 4\alpha\,[(TPS)_{63} + s_{63}] \cdot \left\{\frac{d_{64}}{l_{64}}\, v^{0.5}\right\}$

$$= 4(0.02)(2{,}570{,}000)\left\{\frac{215.1}{42{,}805}\, v^{0.5}\right\}$$

$$= 1008.26$$

EPV(mid-year deaths) $= 4\alpha\,[(TPS)_{63} + 0.5\, S_{63}] \cdot \left\{\frac{r_{63}}{l_{63}}\, v^{0.5}\right\}$

$$= \; 4(0.02)(2{,}485{,}000)\{\, 0.004387\} = 872.14$$

Question 46

(a)

(i) $$ {}_{7.5}E_{57.5:57.5} = v^{7.5}\ ({}_{7.5}p_{57.5})^2 = v^{7.5}\left(\frac{l_{65}}{l_{57.5}}\right)^2 $$

$$ = 1.05^{-7.5}\ \left(\frac{94{,}579.7}{(0.5)(97{,}435.2 + 97{,}195.6)}\right)^2 = 0.6551081 $$

(ii) $$ a^{w}_{57.5} = v\ \frac{l_{58.5}}{l_{57.5}}\ a^{w}_{58.5} = \frac{1}{1.05}\ \frac{(0.5)(97{,}195.6+96{,}929.6)}{(0.5)(97{,}435.2+97{,}195.6)}\ (10.5804) = 10.050395 $$

Alternatively,

$$ \begin{aligned} a^{w}_{57.5} &= {}_{7.5}E_{57.5:57.5}\ \ddot{a}^{(12)}_{\overline{65:65}} + {}_{7.5}E_{57.5}\ (1 - {}_{7.5}p_{57.5})\ddot{a}^{(12)}_{65} \\ &= {}_{7.5}E_{57.5}\ \ddot{a}^{(12)}_{65} + {}_{7.5}E_{57.5:57.5}\left(\ddot{a}^{(12)}_{65} - \ddot{a}^{(12)}_{65:65}\right) \\ &= (0.674057)(13.0807) + (0.6551081)(13.0870 - 11.2158) \\ &= 8.8213838 + 1.0258383 = 10.04722 \end{aligned} $$

where

$$ {}_{7.5}E_{57.5} = 1.05^{-7.5}\ \left(\frac{94{,}579.7}{(0.5)(97{,}435.2 + 97{,}195.6)}\right) = 0.674057 $$

Graders' Comments: *Candidates generally did well on part (i) but struggled with part (ii). Common errors were misapplied recursive annuity formula and using an annuity-due instead of an annuity-immediate. The annuity is deferred to age 65 so there is a payment at t=0.*

(b)

$$ AL = (0.018)(35)(100{,}000)\left\{\frac{w_{57}}{l_{57}}v^{0.5}\ a^{w}_{57.5} + \frac{w_{58}}{l_{57}}v^{1.5}\ a^{w}_{58.5} + \frac{w_{59}}{l_{57}}v^{2.5}\ a^{w}_{59.5}\right\} $$

$$ \begin{aligned} &= (63{,}000)\left\{\frac{1976}{99{,}960.2}1.05^{-0.5}(10.04722) + \frac{1929.9}{99{,}960.2}1.05^{-1.5}(10.5804)\right. \\ &\qquad\left. + \frac{1884.3}{99{,}960.2}1.05^{-2.5}(11.1456)\right\} \\ &= (63{,}000)\{0.193825 + 0.1898566 + 0.1859744\} = 35{,}888.33 \end{aligned} $$

Graders' Comments: *In general, if candidates applied first principles to the contingent benefit they did well. Trying to shoehorn memorized formulas did not work well. Some candidates applied mortality decrements from SULT instead of retirement decrements from Standard Service Table, and many candidates missed that there were 3 possible withdrawal ages.*

(c)

$$NC = (0.018)(100{,}000)\left\{(0.5)\ \frac{w_{57}}{l_{57}} v^{0.5} a^{w}_{57.5} + \frac{w_{58}}{l_{57}} v^{1.5} a^{w}_{58.5} + \frac{w_{59}}{l_{57}} v^{2.5} a^{w}_{59.5}\right\}$$

$$= (1{,}800)\{(0.5)(0.193825) + 0.1898566 + 0.1859744\} = 850.94$$

Or

$$NC = vp_{57} AL_1 + EPV(mid - year - exits) - AL_0$$

$$= 24{,}353.85 + 12{,}35.46 - 35{,}888.33 = 850.94$$

$$vp_{57} AL_1 = (36)(0.018)(100{,}000)\left\{\left(\frac{w_{58}}{l_{57}}\right) v^{1.5} a^{w}_{58.5} + \left(\frac{w_{59}}{l_{57}}\right) v^{2.5} a^{w}_{59.5}\right\}$$

$$= (64{,}800)\{0.1898566 + 0.185974\} = 24{,}353.85$$

$$EPV(mid - year - exits) = (35.5)(0.018)(100{,}000)\left(\frac{w_{57}}{l_{57}}\right) v^{0.5} a^{w}_{57.5}$$

$$= (63{,}900)(0.193825) = 12{,}385.46$$

Graders' Comments: *Many candidates tried to use the PUC shortcut formula for a TUC plan. Those who used the TUC shortcut did not allow for the benefit being contingent on withdrawal.*

(d)

Value of settlement pre-divorce:
$(0.018)(35.5)(100{,}000)a^{w}_{57.5} = (63{,}900)(10.04722) = 642{,}017.36$

Value of settlement post-divorce:
$${}_{7.5}E_{57.5}\ \ddot{a}^{(12)}_{65}\ X \ +\ {}_{7.5}E_{57.5}\ \ddot{a}^{(12)}_{65}\ \frac{X}{3} = (0.674057)(13.0870)\frac{4X}{3} = 11.761845\ X$$

For the two values to be equal, X=54,584.75

Question 47

(a)

F: account balance

$$F = \int_0^{25} c\, S_{35+t}\, e^{\delta(25-t)}\, dt$$

$$F = \int_0^{25} (0.12)\,(50{,}000)\, e^{0.02t}\, e^{0.06(25-t)}\, dt$$

$$= 6000\, e^{0.06(25)} \left(\frac{1 - e^{-0.04(25)}}{0.04}\right) = 424{,}945.17$$

Graders' Comments: *Some candidates treated the salary as an annual lump sum. Also, some confused the projection of the salary between enrollment and payment, with the accumulation from payment until retirement.*

(b)

(i) X: monthly income

$$424{,}945.17 = 12X\left(\ddot{a}^{(12)}_{\overline{10}|} + {}_{10}E_{60}\, \ddot{a}^{(12)}_{70}\right)$$

$$= 12X\left[7.92949 + (0.57864)\left(12.0083 - \tfrac{11}{24}\right)\right]$$

$$= 12X\,[7.92949 + (0.57864)(11.54997)]$$

$$= 12X\,(14.61276)$$

$$\ddot{a}^{(12)}_{\overline{10}|} = \frac{1-v^{10}}{d^{(12)}} = \frac{1-1.05^{-10}}{0.04869} = 7.92949$$

⇨ X = $424{,}945.17/[(12)(14.61276)] = 2423.37$

(ii) Final 1-year salary is

$$\int_{24}^{25} (50{,}000) e^{0.02t} dt = 81{,}617.17$$

$$R = \frac{Pension\ income\ in\ year\ post\ retirement}{Salary\ in\ year\ prior\ to\ retirement} = \frac{29{,}080.42}{81{,}617.17} = 0.3563$$

R=35.63%

Graders' Comments: *Common mistakes were using UDD instead of Woolhouse to calculate the monthly annuity or using an annual annuity factor. Additionally, the replacement ratio compares the first year in retirement to the last year in employment. Many candidates used the rate of salary instead of the salary during the last year.*

(c)

$$\begin{aligned}
750{,}000 &= 12X\left(\ddot{a}^{(12)}_{60:60} + \frac{2}{3}\left(\ddot{a}^{(12)}_{\overline{60:60}} - \ddot{a}^{(12)}_{60:60}\right)\right) \\
&= 12X\left(\ddot{a}^{(12)}_{60:60} + \frac{2}{3}\left(\left(2\ddot{a}^{(12)}_{60} - \ddot{a}^{(12)}_{60:60}\right) - \ddot{a}^{(12)}_{60:60}\right)\right) \\
&= 12X\left(\frac{4}{3}\ddot{a}^{(12)}_{60} - \frac{1}{3}\ddot{a}^{(12)}_{60:60}\right) \\
&= 12X\left(\frac{4}{3}\left(\ddot{a}_{60} - \frac{11}{24}\right) - \frac{1}{3}\left(\ddot{a}_{60:60} - \frac{11}{24}\right)\right) \\
&= 12X\left(\frac{4}{3}\left(14.9041 - \frac{11}{24}\right) - \frac{1}{3}\left(13.2497 - \frac{11}{24}\right)\right) \\
&= 12X\,(14.9972)
\end{aligned}$$

X=4167.44

Graders' Comments: *There were many ways to calculate the annuity values in this part, but many candidates are not comfortable with annuities whose payment depends on a two-life status.*

(d)

(i) (1) (4167.44) + 1000 = 5167.44
(ii) (2/3)(4167.44) + 1000 = 3778.29
(iii) (2/3)(4167.44) + 500 = 3278.29

(e)

$$\begin{aligned}
750{,}000 &= 12X\left(\frac{4}{3}\ddot{a}^{(12)}_{60} - \frac{1}{3}\ddot{a}^{(12)}_{60:60}\right) \\
&\quad - 12\,(1000)\left(\tfrac{4}{3}\ddot{a}^{(12)}_{65} - \tfrac{1}{3}\ddot{a}^{(12)}_{65:65}\right){}_5E_{60:60} \qquad \text{(i.e. both alive at 65)} \\
&\quad - 12\,(1000)\,2\left(\tfrac{2}{3}\ddot{a}^{(12)}_{65}\right){}_5E_{60}\,({}_5q_{60}) \qquad \text{(i.e. only one alive)}
\end{aligned}$$

where $\frac{4}{3}\ddot{a}^{(12)}_{60} - \frac{1}{3}\ddot{a}^{(12)}_{60:60} = 14.9972$ from part c).

$$\begin{aligned}
750{,}000 &= 12X\,(14.9972) \\
&\quad - 12\,(1000)\left(\tfrac{4}{3}(13.5498 - \tfrac{11}{24}) - \tfrac{1}{3}\left(11.6831 - \tfrac{11}{24}\right)\right)(0.75057) \\
&\quad - 12\,(1000)\,2\left(\tfrac{2}{3}(13.5498 - \tfrac{11}{24})\right)(0.76687)(0.02126) \\
&= 12X\,(14.9972) - 123{,}516.35 - 3414.95
\end{aligned}$$

$=>$ $X = 4872.74$

$${}_5E_{60:60} = ({}_5E_{60})^2(1+i)^5 = (0.76687)^2(1.05)^5 = 0.75057$$
$${}_5E_{60:60} = v^5({}_5p_{60})^2 = (1.05)^{-5}(\tfrac{94{,}579.7}{96{,}634.1})^2 = (1.05)^{-5}(0.97874)^2 = 0.75057$$

$${}_5q_{60} = 1 - {}_5p_{60} = 1 - \frac{94{,}579.7}{96{,}634.1} = 1 - 0.97874 = 0.02126$$

Graders' Comments: *Few attempted this part and very few got it right. The key is to consider all the possible outcomes when the couple (jointly or singly) reaches 65..*

Question 48

(a)

(i) The EPV at age 63.5 is

$$EPV = 12.5 \times 0.016 \times FAS_{63.5} \times (1 - 18 \times 0.004) \times \ddot{a}_{63.5}^{(12)}$$
where $FAS_{63.5} = 0.5 \times 72{,}100 + 0.5 \times 70{,}000 = 71{,}050$
$$EPV = 13{,}186.88 \times 13.5139 = 178{,}206.18$$

(ii) The EPV at 1 Jan 2022 of the mid-year exits is

$$\frac{r_{63}}{l_{63}} v^{0.5}(178{,}206.18) = \frac{4{,}515.2}{47{,}579.3} 1.05^{-0.5}(178{,}206.18)$$
$= 16{,}503.92$ (or $16{,}503.90$ if using $178{,}206$)

(b)

(i) The projected actuarial liability (AL) at 1/1/2023 is the value at 1/1/2023 of the exit benefits for ages 64.5 and 65, based on 13 years of past service.

The AL at 1/1/2022 is the value at 1/1/2022 of the exit benefits for ages 63.5, 64.5 and 65, based on 12 years of past service. This gives us

$$AL = \frac{12}{12.5} \times 16{,}503.92 + \frac{12}{13} \times v \times \frac{l_{64}}{l_{63}} \times 191{,}309$$

$$= \frac{12}{12.5} \times 16{,}503.92 + \frac{12}{13} \times 1.05^{-1} \times \frac{42{,}805.0}{47{,}579.3} \times 191{,}309$$

$$= 15{,}843.76 + 151{,}307.50 = 167{,}151.26$$

(ii) The normal cost is the value at 1/1/2022 of the mid-year exits plus the value at 1/1/2022 of the projected 1/1/2023 AL minus the 1/1/2022 AL. That is

$$NC = 16{,}503.92 + 191{,}309 \times \frac{l_{64}}{l_{63}} \times v - 167{,}151.26 = 13{,}269.11$$

$$AL = \frac{0.5}{12.5} \times 16{,}503.92 + \frac{1}{13} \times v \times \frac{l_{64}}{l_{63}} \times 191{,}309$$

$$= \frac{0.5}{12.5} \times 16{,}503.92 + \frac{1}{13} \times 1.05^{-1} \times \frac{42{,}805.0}{47{,}579.3} \times 191{,}309$$

$$= 660.16 + 12{,}608.96 = 13{,}269.11$$

(c)

(i) The AL using TUC will be <u>smaller</u> than the PUC AL.

- The TUC AL does not project the FAS to retirement, using the current FAS instead. As salaries are increasing, this means that the TUC AL is always less than the PUC AL, except at the very start of the member's employment, when service $n = 0$, so both ALs are 0, and at the very last retirement date, when the FAS is the same under both PUC and TUC.

(ii) The TUC NC will be <u>greater</u> than the PUC NC.

- The TUC normal cost each year starts below the PUC, but around 2/3 through the max employment period, the lines cross. In B's case, as they are near to retirement, the TUC NC will be greater than the PUC NC.

(d)

$$(AL_{2022} + NC_{2022})(1.051) = (167{,}151.26 + 13{,}269.11)(1.051) = 189{,}621.81$$

$$13 \times 0.016 \times 72{,}100 \times (1 - 12 \times 0.004) \times 13.3735$$
$$= 14{,}276.95 \times 13.3735 = 190{,}932.84$$

$$189{,}621.81 - 190{,}932.84 = -1311.03$$

The plan made a loss of 1311 from the early retirement.

Graders' Comments: *Very few candidates attempted this part. When a question is unfamiliar, returning to first principles is usually the best strategy.*

Question 49

(a) $AV_t = \left(AV\right)\{t-1\} + P_t - E_t - COI_t\Big)\left(1+i_t^c\right)$

where $COI_t = ADB_t \times q_{x+t-1} \times v_{i_q}$, $ADB_t = 150,000$, and $i^q = 0.04$;

$\Rightarrow AV_3 = \left(8166 + 4800(0.95) - 377.88\right)(1.04) = 12,842$

The total DV in year 3 is $DB_3 = ADB_3 + AV_3 = 162,842$

(b) The profit test table is:

Year, t	AV_{t-1}	P_t	E_t	I_t	EDB_t	ESV_t	EAVt	Pr_t
0*	0		800					-800.00
1	0	4800	1200	288	230.47	114.528	3277.377	265.63
2	3647	4800	336	649	237.25	1331.2	6523.001	668.43
3	8166	4800	336	1010	244.26	12822.74		573.40

AV_{t-1} is the account value brought forward.

P_t is the premium paid at the start of the t-th year.

E_t represents the expenses incurred at the start of the *t*th year.

$I_t = \left(AV_{t-1} + P_t - E_t\right)(0.08)$ is the interest earned during the *t*th year, for a policy in force at the start of the year.

$EDB_t = q_{x+t-1}^{(d)}\left(ADB_t + AV_t\right) = 0.0015(150,000 + AV_t)$

$ESV_t = q_{x+t-1}^{(w)}\left(AV_t - SC_t\right)^+$ where $q_{35}^{(w)} = 0.1\left(1 - q_{35}^{(d)}\right)$; $q_{36}^{(w)} = 0.2\left(1 - q_{36}^{(d)}\right)$

and $q_{37}^{(w)} = \left(1 - q_{37}^{(d)}\right)$. SC_t is the surrender charge in year *t*.

$EAV_t = p_{x+t-1}^{(\tau)} AV_t$ where $p_{x+t-1}^{(\tau)} = 1 - q_{x+t-1}^{(d)} - q_{x+t-1}^{(w)}$

Pr_t is the profit vector.

(c)

Year, t	Pr_t	Π_t
0	-800.00	-800.00
1	265.63	265.63
2	668.43	600.68
3	573.40	411.61

Where $\Pi_0 = \mathrm{Pr}_0$, and $\Pi_t = {}_{t-1}p_x^{(\tau)}\ \mathrm{Pr}_t$.

$\Rightarrow NPV = -800 + 265.63v + 600.68v^2 + 411.61v^3$ at 10%

$= 247.2$

(d) The profit margin is

$$\frac{NPV}{EPV\ \text{Premiums at 10\%}} = \frac{247.2}{4800\left(1 + p_x^{(\tau)} v + {}_2 p_x^{(\tau)} v^2\right)} = \frac{247.2}{11569.0}$$

$$= 2.14\%$$

Question 50

(a) For each year, we assume first that the corridor factor does not apply, and then check to see whether the Death Benefit: Account Value ratio is sufficient. For the first year we have

$$AV_1 = \left(50{,}000(0.8) - 75 - (0.025)(100{,}000 - AV_1)v_{4.5\%}\right)(1.065)$$

$$\Rightarrow AV_1(1 - 0.025v_{4.5\%}(1.065)) = 39{,}972$$

$$\Rightarrow AV_1 \;=\; 41{,}017$$

Check corridor factor: $\dfrac{\text{DB}}{AV_1} = \dfrac{100{,}000}{41{,}017} = 2.44 > CF_1$

Since the ratio of the DB to the AV is greater than the corridor factor, not adjustment to the DB is required.

For the second year, we have

$$AV_2 = \left(41{,}017 + 50{,}000(0.92) - 75 - (0.03)(100{,}000 - AV_2)v_{4.5\%}\right)(1.0575)$$

$$\Rightarrow AV_2(1 - 0.03v_{4.5\%}(1.0575)) = 88{,}906$$

$$\Rightarrow AV_2 \;=\; 91{,}689$$

Check corridor factor: $\dfrac{DB}{AV_2} = \dfrac{100{,}000}{91{,}689} = 1.09 < CF_1$

The ratio is too small, and the corridor factor applies. We must recalculate the account value.

$$AV_2 = \left(41{,}017 + 50{,}000(0.92) - 75 - (0.03)(0.4 \times AV_2)v_{4.5\%}\right)(1.0575)$$

$$\Rightarrow AV_2(1 + 0.03(0.4)v_{4.5\%}(1.0575)) = 91{,}9416$$

$$\Rightarrow AV_2 \;=\; 90{,}838$$

Graders' Comments*: The most common errors were ignoring the corridor factor, or assuming the corridor factor applied in both years.*

(b) The EPV of the annuity is

$$4Q\left(\ddot{a}^{(4)}_{\overline{10}|} + {}_{10}E_{60:70}\left(\ddot{a}^{(4)}_{70} + 0.6\,\ddot{a}^{(4)}_{70|80}\right) + ({}_{10}E_{70})\,({}_{10}q_{60})\left(0.6\ddot{a}^{(4)}_{80}\right) + ({}_{10}E_{60})\,({}_{10}q_{70})\,\ddot{a}^{(4)}_{70}\right)$$

$$\ddot{a}^{(4)}_{\overline{10}|} = \frac{1-v^{10}}{d^{(4)}} = 7.96219$$

$$\ddot{a}^{(4)}_{70} = \ddot{a}_{70} - \frac{3}{8} = 11.6333; \quad \ddot{a}^{(4)}_{70:80} = \ddot{a}_{70:80} - \frac{3}{8} = 7.3458$$

$$\ddot{a}^{(4)}_{80} = \ddot{a}_{80} - \frac{3}{8} = 8.1734; \quad \ddot{a}^{(4)}_{70|80} = \ddot{a}^{(4)}_{80} - \ddot{a}^{(4)}_{70:80} = 0.82761$$

$${}_{10}q_{60} = 0.05745; \quad {}_{10}q_{70} = 0.16935$$

$${}_{10}E_{60} = 0.57864; \quad {}_{10}E_{70} = 0.50994; \quad {}_{10}E_{60:70} = {}_{10}E_{60}\,{}_{10}p_{70} = 0.48064$$

Thus, $EPV = 4Q(15.0759)$

Set this equal to AV_2 to get $Q = 1506.34$.

***Graders' Comments**: Most candidates used Q instead of 4Q throughout.*

(c) ${}_{10}V = 4Q\ddot{a}_{70}^{(4)} = 70095$

(d) It is uncommon to offer surrender values for annuities because of adverse selection. Lives who are in poor health are more likely to surrender, reducing the pool available for survivors. The objective of an annuity pool is that any excess funds arising from early deaths are used to support the continuing payments to the surviving lives.

***Graders' Comments**: Most candidates seemed unaware that traditional annuities cannot generally be surrendered, and few mentioned the adverse selection problem.*

Question 51

(a) $AV_2 = \left(AV_1 + 0.9P_2 - 10 - 200v \right)(1.06) = 918.3$

(b) Now $AV_2 = \left(165 + 0.9P_2 - 10\right)(1.06) - 200 = 0.954P_2 - 35.7$

$$AV_3 = \left(AV_2 + 0.9P_3 - 10 \right)(1.06) - 300$$
$$= \left(0.954P_2 - 35.7 + 0.9P_3 - 10\right)(1.06) - 300$$
$$= 1.0112P_2 + 0.954P_3 - 348.4$$
$$\Rightarrow a = 1.0112; b = 0.954; c = -348.4$$

(c) There are four possible outcomes for AV_3, and hence for the death benefit and surrender values. Note that the surrender value cannot be negative.

P_1	P_2	AV_3	DB_3	SV_3	Prob
1000	1000	1616.8	101,616.8	1516.8	0.36
1000	200	853.6	100,853.6	753.6	0.24
200	1000	807.8	100,807.8	707.8	0.08
200	200	44.6	100,044.6	0	0.32

So

(i) $\mathrm{E}\left[DB_3\right] = 101,616.8 \times 0.36 + \ldots + 100,044.6 \times 0.32 = 100,866$

(ii) $\mathrm{E}\left[SV_3\right] = 1516.8 \times 0.36 + \ldots + 0 \times 0.32 = 783.5$

(d) Let $AV_{10}^{(1)} = 5114$ denote the account value at time 10 if all the premiums are paid, $AV_{10}^{(2)}$, and $AV_{10}^{(3)}$, denote the account values if the 3rd premium and the 10th premium, respectively, are omitted. Then

$$AV_{10}^{(2)} = AV_{10}^{(1)} - 1000(0.9)(1.06)^8 = 3679.5$$
$$AV_{10}^{(3)} = AV_{10}^{(1)} - 1000(0.95)(1.06) = 4107.0$$
$$\Rightarrow \mathrm{E}\left[AV_{10}\right] = 0.5 \times 3679.5 + 0.5 \times 4107.0 = 3893.3$$

Graders' Comments: *In part (c)(ii), many candidates subtracted the surrender charge from* $\mathrm{E}\left[AV_3\right]$, *which does not work because of the floor of 0 on the surrender value. Many candidates omitted part (d), but those who attempted it scored well.*

Question 52

(a) $AV_1 = \left(2500(0.95) - 0.005(100,000)v_{6\%} - 30\right)(1.06) = 1985.70$

$\Rightarrow CV_1 = 0.2AV_1 = 397.1$

(b) $\mathrm{Pr}_1 = \left(P - E\right)(1 + i^e) - EDB_1 - ESV_1 - EAV_1$

$P = 2500; \quad E = 120; \quad i^e = 0.11;$

$EDB_1 = 0.004\left(100,000 + AV_1 + 200\right) = 408.7$

$ESV_1 = 0.1(0.996)\left(CV_1 + 100\right) = 49.5$

$EAV_1 = 0.996(0.9)AV_1 = 1780.0$

$\Rightarrow \mathrm{Pr}_1 = 403.6$

Graders Comments: *Part (a) was done well, although a few assumed that this was a Type A policy, when it is a Type B (fixed additional death benefit). In part (b), the most common errors were in calculating the dependent withdrawal and survival probabilities, failing to include the account value in EDB, and using the contract expense assumptions instead of the profit test expense assumptions.*

(c) $NPV(1) = \mathrm{Pr}_0 + \mathrm{Pr}_1\, v_{14\%} = -200 + 403.6v = 154.0$

(d) Let * denote values under the revised assumptions. Then we have

$NPV = \mathrm{Pr}_0 + \mathrm{Pr}_1\, v + p_x^{(\tau)}\, \mathrm{Pr}_2\, v^2 + {}_2p_x^{(\tau)}\, \mathrm{Pr}_3\, v^3 + \ldots \quad \text{at } i = 0.14$

$\mathrm{Pr}_t^* = \mathrm{Pr}_t \ \text{ for } t = 0, 2, 3, \ldots$

$EDB_1^* = EDB_1 = 408.7$

$ESV_1^* = 0.2(0.996)497.1 = 99.0$

$EAV_1^* = 0.996(0.8)AV_1 = 1582.2$

$\Rightarrow \mathrm{Pr}_1^* = \left(2500 - 120\right)(1.11) - 408.7 - 99.0 - 1582.2 = 551.9$

Also, for $k = 1, 2, \ldots, \quad {}_kp_x^{(\tau)*} = \dfrac{0.8}{0.9}\, {}_kp_x^{(\tau)}$

$$\Rightarrow NPV^* = \left(NPV - \mathrm{Pr}_0 - \mathrm{Pr}_1\, v\right)\frac{0.8}{0.9} + \mathrm{Pr}_0 + \mathrm{Pr}_1^*\, v$$

$$= \left(2000 - 154\right)\frac{0.8}{0.9} - 200 + 551.9v = 1925.0$$

Graders' Comments: *The change in assumptions impacts Pr$_1$, which most candidates correctly allowed for, but it also changes the survival probabilities used for all the profit signature values from the 2nd contract year, which most candidates did not allow for.*

Question 53

(a) Universal life is regulated as an insurance product. The corridor factor ensures that the insurance benefit is significant throughout the term of the contract, differentiating the policy from a pure investment contract.

(b)

(i) $AV_1 = \left(15,000(0.99) - CoI\right)(1.05)$

$$CoI = 1.2q_{50}\left(100,000 - AV_1\right)v_{4\%} = 139.62 - 0.0013962AV_1$$

$$\Rightarrow AV_1 = \frac{15,445.9}{0.998534} = 15,468.6$$

$$\Rightarrow CoI = 118.02$$

(ii) $CoI = 1.2q_{50}\left(1.2AV_1\right)v_{4\%} = 0.001675AV_1$

$$\Rightarrow AV_1 = \frac{15,592.5}{1.001759} = 15,565.12$$

$$\Rightarrow CoI = 26.07$$

(iii) $CoI = \max\left(118.02,\ 26.07\right) = 118.02$

(c) $AV_1 = 15,468.6$ from (b)(i)

$$ADB_1 = DB_1 - AV_1 = 100,000 - 15,486.6 = 84,513.4$$

$$CV_1 = 15,486.6 - 0.05(15,000) = 14,736.6$$

(d) The value of the term insurance is

$$100,000A^{\;1}_{70:\overline{20|}} = 100,000\left(A_{70:\overline{20|}} - {}_{20}E_{70}\right) = 29,778$$

$\Rightarrow$ The reserve is $29,778 - 20,000 = 9,778.$

Graders' Comments: *Many candidates memorized the CoI formulas for part (b). If the formulas used were correct, full marks were awarded, as the question did not ask for a derivation. Some candidates answered the question as if the contract were a Type B. This earned little credit as it significantly simplifies the work, and because understanding the difference between Type A and Type B contracts is important.*

Question 54

(a)

$$AV_1 = \left(0.4(5000) - 200 - 0.006(100000)v_{5\%}\right)(1.05) = 1290.0$$

Graders' Comments: *A number of candidates incorrectly used information from the `Assumptions' table in this part. It is important in UL to understand the different roles of the policy terms and conditions (used to project account values) and the valuation assumptions (used to determine profitability).*

(b)

$$NPV(3) = -1206v_{12\%} + 374\,p_{60}^{(\tau)}\,v_{12\%}^2 + 400\;{}_2p_{60}^{(\tau)}\,v_{12\%}^3$$

$$p_{60}^{(\tau)} = 1 - 0.004 - 0.1 = 0.8960; \quad {}_2p_{60}^{(\tau)} = 0.8960(0.9450) = 0.8467$$

$$\Rightarrow NPV(3) = -568.6$$

(c)

(i) Revised profit test table for year $t = 2$:

${}_{t-1}V$	P	E	I	EDB	ESV	$E\,{}_tV$	Pr_t
690	1,000	60	97.8	507.5	65	1,228.5	−73.2

$${}_1V = 1290 - 600 = 690$$

$$\textit{Premium},\ P = 1000$$

$$\textit{Expenses},\ E = 0.05P + 10 = 60$$

$$\textit{Interest},\ I = 0.06\left(690 + 1000 - 60\right) = 97.80$$

$$EDB = q_{61}^{(d)} \times (100000 + AV_2) = 507.5$$

$$ESV = q_{61}^{(w)} \times (1500 - 200) = 65$$

$$E_tV = p_{61}^{(\tau)} \times (1500 - 200) = 1228.5$$

(ii) In year 2 the beginning of year reserve, brought forward with interest during year 2 is much lower using the cash value. The end year reserve is also reduced, but the change is smaller as the surrender charge is lower. Hence, the surplus emerging in year 2 decreases.

(iii) Overall the total present value of profits will increase, because the profits are released earlier when reserves are smaller. This increases the NPV whenever the risk discount rate is greater than the assumed earned rate.

Graders' Comments: *Many omitted part (c). Of the candidates who attempted it, many answered (c)(i) correctly or with a minor error (such as omitting AV_2 from the expected cost of death benefit).*

Parts (c)(ii) and (iii) proved to be challenging conceptual questions. A few strong candidates correctly answered (c)(ii), and a handful correctly (and impressively) answered (c)(iii). Some candidates offered one-word answers (`increase' or `decrease'), which received no credit.

Question 55

Note: This question is on a new topic. It is particularly suited to the use of Excel for calculations. The candidate would be expected to include all the detail given here, at a minimum, for full credit. It will not be sufficient to write down the numerical answers without explanation. Excel workbooks will not be submitted or graded.

(a)

t	F_t	MC_t	F_{t+1}
0	5000.0	125.0	4192.5
1	4192.5	104.8	3883.3
2	3883.3	97.1	3937.7

Where $\mathrm{MC}_t = 0.025F_t$ and $F_{t+1} = (F_t - MC_t)(1+R_t)$, and R_t is the return on assets in the t-th year.

(b)

t	Income	Expenses	Interest	EDB_t	EMB_t	Pr_t
0*		200.0				-200.0
1	475.0	100.0	18.8	40.4		353.4
2	104.8	71.9	1.6	67.0		-32.5
3	97.1	68.8	1.4	74.4	988.0	-1032.7

Explanation:

- The income is the MC from part (a), plus 350 policy fee in the first year.
- Expenses at time 0 are the 200 pre-contract costs; in year 1 the expenses are 2% of F_0; in year 2 and 3 the expenses are 1% of F_t. plus 30.
- Interest is 5% of income – expenses.
- The expected death benefit cost, EDB_t, is $q_{x+t-1}(5000 - F_t)^+$
- The expected maturity benefit cost, EMB_t is 0 in the first two years. In the third year is is $p_{x+2}(5000 - F_3)^+$
- The Profit vector value for the t-th year is
  $\mathrm{Pr}_t = \mathrm{Income}_t - \mathrm{Expenses}_t + \mathrm{Interest}_t - \mathrm{EDB}_t - \mathrm{EMB}_t$

(c)

t	Pr_t	${}_{t-1}p_x$	Π_t	NPV(t)
0	-200.0		-200.0	-200
1	353.4	1	353.4	121.2
2	-32.5	0.95	-30.8	95.8
3	-1032.7	0.893	-922.2	-597.1

Explanation:

- Pr_t is the profit vector from (b)
- ${}_{t-1}p_x$ is the survival probability to the start of the t-th year

- The profit signature at t is $\Pi_t = {}_{t-1}p_x \Pr_t$ for $t \geq 1$; $\Pi_0 = \Pr_0$.
- NPV(t) is the partial NPV:

$NPV(0) = \Pi_0$;

$NPV(t) = NPV(t-1) + \Pi_t \, v_{\text{rdr}}^t$

Where the discount factor uses the risk discount rate of 10%.

The NPV is NPV(3)= –597.1.

(d) The Black Scholes Price for a put option on a premium of P with a guarantee of kP, is

$$\text{BSP}(t) = P\left(ke^{-r(T-t)}\Phi\left(-d_2\right) - \xi\,\Phi\left(-d_1\right)\right)$$

where ξ is the expense deduction factor, and

$$d_1 = \frac{\log(\xi/k) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}.$$

In this case:

$$P = 1,\ k = 1,\ \xi = \left(1 - 0.025\right)^3 = 0.92686$$

$$d_1 = \frac{\log(0.92686) + (0.04 + 0.25^2/2)3}{0.25\sqrt{3}} = 0.318228$$

$$d_2 = 0.318228 - 0.25\sqrt{3} = -0.114785$$

$$\Rightarrow\ \text{BSP} = e^{-3(0.04)}\Phi\left(0.114785\right) - 0.92686\,\Phi\left(-0.318228\right) = 0.13627$$

(e) The GMDB option cost is

$$q_x(446.7) + {}_{1|}q_x(590.8) + {}_{2|}q_x\left(5000 \times 0.13627\right) = 98.60$$

as $q_x = 0.05,\ {}_{1|}q_x = 0.95 \times 0.06 = 0.057,$ and ${}_{2|}q_x = 0.95 \times 0.94 \times 0.07 = 0.0625$

The GMMB cost is

$${}_3p_x \times 5000 \times 0.13627 = 565.85$$

So the total cost is 664.45

(f)

t	Income	Expenses	Interest	EDB_t	EMB_t	$\Pr_t$
0*		200.0				-200.0
1	475.0	765.0	-14.5			-304.5
2	104.8	71.9	1.6			34.5
3	97.1	68.8	1.4			29.7

$$\text{NPV} = -200 - 304.5v + p_x(34.5)v^2 + {}_2p_x(29.7)v^3 = -429.8, \text{ at } 10\%$$

(g)

Advantage: Hedging the guarantees limits the downside risk; we see from this stress test that we are replacing the uncertain and potentially very serious future loss from guarantees, with the certain cost of buying the options.

Disadvantage: The cost of the options is substantial. The income for this policy is 350 from the policy fee, plus around 1.5% of the fund annually (the management charge minus the fund management costs) , which even on optimistic assumptions is unlikely to be greater than 550 in total. This is insufficient to cover the hedge costs. (This is a sign that the guarantees are under-priced, rather than an argument against hedging.).

Question 56

(a) $\pi(0) = \mathrm{E}_0^Q\left[e^{-10r}\left(kP - F_{10}\right)^+ \ \mathrm{I}(T_x > 10)\right]$ where P is the premium, k is the proportion of premium guaranteed under the GMMB, r is the risk free rate, F_{10} is the fund value at time 10, and $\mathrm{I}(T_x > 10)$ is an indicator random variable taking the value 1 if (x) survives 10 years, 0 otherwise. The Q superscript indicates that we are taking expectations under the risk neutral measure, and the 0 subscript indicates that we are valuing based on the information available at time 0.

In our case, $k = 1$, and $F_t = P(0.95)e^{-12(0.002)t} S_t$, for $t = 0, \tfrac{1}{12}, \tfrac{2}{12}, \tfrac{3}{12}, \cdots, 10.$

We assume that the T_x and F_t random variables are independent, and that the Q and P measure expectations for the future lifetime are the same. Then

$\mathrm{E}_0^Q\left[I(T_x > 10)\right] = \Pr\left[T_x > 10\right] = {}_{10}p_x.$

Also let $\xi = 0.95\left(e^{-120*0.002}\right) \Rightarrow F_{10} = P\xi S_{10}$

$$\text{Then } \pi(0) = {}_{10}p_x \ \mathrm{E}_0^Q\left[e^{-10r}\left(P - P\xi S_{10}\right)\right]$$

$$= {}_{10}p_x \ P\,\xi\,\mathrm{E}_0^Q\left[e^{-10r}\left(\frac{1}{\xi} - S_{10}\right)\right]$$

$$= {}_{10}p_x \ P\,\xi\,\mathrm{BSP}\left(\tfrac{1}{\xi}\right) \text{ as required, with } \xi = 0.74730 \text{ and } K^* = \xi^{-1}.$$

(b)

$${}_{10}p_{60} = 0.94255; \quad P = 100,000; \quad \xi = 0.747230$$

$${}_{10}p_{60} P\xi BSP\left(\tfrac{1}{\xi}\right) = {}_{10}p_{60} P\xi\left(\frac{1}{\xi}e^{-10r}\Phi\left(-d_2\right) - \Phi\left(-d_1\right)\right)$$

$$= {}_{10}p_{60} P\left(e^{-10r}\Phi\left(-d_2\right) - \xi\Phi\left(-d_1\right)\right)$$

$$d_1 = \frac{\log(\xi) + (r + \sigma^2/2)(10)}{\sigma\sqrt{10}} = 0.53279; \quad d_2 = d_1 - \sigma\sqrt{10} = -0.25778$$

$$\Rightarrow \pi(0) = 0.94255(100,000)\left(e^{-10\times 0.04}\Phi\left(0.25778\right) - 0.74723\Phi\left(-0.53279\right)\right)$$

$$= 0.94255(100,000)(0.181325) = 17,091$$

The normal cdf values are found using Excel.

(c) The bond part of the hedge at $t = 0$ is ${}_{10}p_{60}Pe^{-10(0.04)}\Phi\left(-d_2\right) = 38017$.

The stock part of the hedge at $t = 0$ is $-{}_{10}p_{60}P\xi\ \Phi\left(-d_1\right) = -20926$.

After one month, the bond part of the hedge is worth $38,017e^{0.04/12} = 38,144$

After one month, the stock part of the hedge is worth $-20,926(0.97) = -20,298$

The total hedge value before rebalancing is 17,846.

(d) Let c denote the annual rate of payment, and let $m = 12 \times 0.002$ denote the annual management charge rate. The equation of value, under the Q measure, is

$$17,091 = \mathrm{E}_0^Q\left[\frac{c}{12} F_{1/12} e^{-r/12}\ I\left(T_{60} > 1/12\right)\right] + \mathrm{E}_0^Q\left[\frac{c}{12} F_{2/12} e^{-2r/12}\ I\left(T_{60} > 2/12\right)\right] + \ldots$$

$$\ldots + \mathrm{E}_0^Q\left[\frac{c}{12} F_{9\,11/12} e^{-9\,11/12 \times r}\ I\left(T_{60} > 9\,11/12\right)\right] + \mathrm{E}_0^Q\left[\frac{c}{12} F_{10} e^{-10r}\ I\left(T_{60} > 10\right)\right]$$

$$= \left(\frac{c}{12}\right) P(0.95)\left\{ {}_{1/12}p_{60} \mathrm{E}_0^Q\left[S_{1/12} e^{-r/12} e^{-m/12}\right] + {}_{2/12}p_{60} \mathrm{E}_0^Q\left[S_{2/12} e^{-2r/12} e^{-2m/12}\right] + \cdots\right.$$

$$\left. + \cdots + {}_{9\,11/12}p_{60} \mathrm{E}_0^Q\left[S_{9\,11/12} e^{-9\,11/12 \times r} e^{-9\,11/12 \times m}\right] + {}_{10}p_{60} \mathrm{E}_0^Q\left[S_{10} e^{-10r} e^{-10m}\right]\right\}$$

Note that under the risk neutral measure, $\mathrm{E}_0^Q\left[S_t e^{-rt}\right] = S_0$, and in this case $S_0 = 1$, so we have

$$17,091 = \frac{c}{12}(0.95) P\left\{ {}_{1/12}p_{60}\, e^{-m/12} + {}_{2/12}p_{60}\, e^{-2m/12} + {}_{3/12}p_{60}\, e^{-3m/12} + \cdots + {}_{10}p_{60} e^{-10m}\right\}$$

$$= \frac{c}{12}(0.95) P\left(12 a^{(12)}_{60:\overline{10|}}\right) \text{ at } i = e^m - 1 = 2.429\%$$

$$\Rightarrow c = \frac{17,091}{0.95 P a^{(12)}_{60:\overline{10|}}} = 0.0207$$

The monthly rate of deduction is thus $\frac{0.0207}{12} = 0.0017$.

Question 57

This question is an Excel question. Please see the ALTAM Sample Excel Solution file for details. The content of the Excel file for this question is provided below for reference.

(a)

Time (t)	μ_{50+t}^{01}	μ_{50+t}^{02}	μ_{50+t}^{12}	${}_tp_{50}^{00}$	${}_tp_{50}^{01}$	${}_tp_{50}^{02}$
0	0.0200	0.0012	0.0023	**1.0000**	**0.0000**	**0.0000**
0.083	0.0200	0.0012	0.0023	**0.9982**	**0.0017**	**0.0001**
0.167	0.0200	0.0012	0.0023	**0.9965**	**0.0033**	**0.0002**
0.250	0.0200	0.0012	0.0024	**0.9947**	**0.0050**	**0.0003**
0.333	0.0200	0.0012	0.0024	**0.9930**	**0.0066**	**0.0004**
0.417	0.0200	0.0012	0.0024	**0.9912**	**0.0083**	**0.0005**
0.500	0.0200	0.0012	0.0024	**0.9895**	**0.0100**	**0.0006**
0.583	0.0200	0.0012	0.0024	**0.9877**	**0.0116**	**0.0007**
0.667	0.0200	0.0012	0.0025	**0.9860**	**0.0132**	**0.0008**
0.750	0.0200	0.0012	0.0025	**0.9842**	**0.0149**	**0.0009**
0.833	0.0200	0.0012	0.0025	**0.9825**	**0.0165**	**0.0010**
0.917	0.0200	0.0013	0.0025	**0.9807**	**0.0182**	**0.0011**
1.000	0.0200	0.0013	0.0025	**0.9790**	**0.0198**	**0.0012**
1.083	0.0200	0.0013	0.0026	**0.9773**	**0.0214**	**0.0013**
1.167	0.0200	0.0013	0.0026	**0.9755**	**0.0230**	**0.0014**
1.250	0.0200	0.0013	0.0026	**0.9738**	**0.0247**	**0.0015**
1.333	0.0200	0.0013	0.0026	**0.9721**	**0.0263**	**0.0017**
1.417	0.0200	0.0013	0.0026	**0.9703**	**0.0279**	**0.0018**
1.500	0.0200	0.0013	0.0027	**0.9686**	**0.0295**	**0.0019**
1.583	0.0200	0.0013	0.0027	**0.9669**	**0.0311**	**0.0020**
1.667	0.0200	0.0014	0.0027	**0.9652**	**0.0327**	**0.0021**
1.750	0.0200	0.0014	0.0027	**0.9635**	**0.0343**	**0.0022**
1.833	0.0200	0.0014	0.0028	**0.9617**	**0.0359**	**0.0023**
1.917	0.0200	0.0014	0.0028	**0.9600**	**0.0375**	**0.0025**
2.000	0.0200	0.0014	0.0028	**0.9583**	**0.0391**	**0.0026**
2.083	0.0200	0.0014	0.0028	**0.9566**	**0.0407**	**0.0027**
2.167	0.0200	0.0014	0.0028	**0.9549**	**0.0423**	**0.0028**
2.250	0.0200	0.0014	0.0029	**0.9532**	**0.0439**	**0.0029**
2.333	0.0200	0.0014	0.0029	**0.9515**	**0.0454**	**0.0031**
2.417	0.0200	0.0015	0.0029	**0.9498**	**0.0470**	**0.0032**
2.500	0.0200	0.0015	0.0029	**0.9481**	**0.0486**	**0.0033**
2.583	0.0200	0.0015	0.0030	**0.9464**	**0.0501**	**0.0034**
2.667	0.0200	0.0015	0.0030	**0.9447**	**0.0517**	**0.0036**
2.750	0.0200	0.0015	0.0030	**0.9430**	**0.0533**	**0.0037**
2.833	0.0200	0.0015	0.0030	**0.9413**	**0.0548**	**0.0038**

Time (t)	μ^{01}_{50+t}	μ^{02}_{50+t}	μ^{12}_{50+t}	${}_t p^{00}_{50}$	${}_t p^{01}_{50}$	${}_t p^{02}_{50}$
2.917	0.0200	0.0015	0.0031	**0.9396**	**0.0564**	**0.0040**
3.000	0.0200	0.0015	0.0031	**0.9380**	**0.0579**	**0.0041**
3.083	0.0200	0.0016	0.0031	**0.9363**	**0.0595**	**0.0042**
3.167	0.0200	0.0016	0.0031	**0.9346**	**0.0610**	**0.0044**
3.250	0.0200	0.0016	0.0032	**0.9329**	**0.0626**	**0.0045**
3.333	0.0200	0.0016	0.0032	**0.9312**	**0.0641**	**0.0047**
3.417	0.0200	0.0016	0.0032	**0.9296**	**0.0656**	**0.0048**
3.500	0.0200	0.0016	0.0032	**0.9279**	**0.0672**	**0.0049**
3.583	0.0200	0.0016	0.0033	**0.9262**	**0.0687**	**0.0051**
3.667	0.0200	0.0017	0.0033	**0.9245**	**0.0702**	**0.0052**
3.750	0.0200	0.0017	0.0033	**0.9229**	**0.0718**	**0.0054**
3.833	0.0200	0.0017	0.0034	**0.9212**	**0.0733**	**0.0055**
3.917	0.0200	0.0017	0.0034	**0.9195**	**0.0748**	**0.0057**
4.000	0.0200	0.0017	0.0034	**0.9179**	**0.0763**	**0.0058**
4.083	0.0200	0.0017	0.0034	**0.9162**	**0.0778**	**0.0060**
4.167	0.0200	0.0017	0.0035	**0.9146**	**0.0793**	**0.0061**
4.250	0.0200	0.0018	0.0035	**0.9129**	**0.0808**	**0.0063**
4.333	0.0200	0.0018	0.0035	**0.9112**	**0.0823**	**0.0064**
4.417	0.0200	0.0018	0.0036	**0.9096**	**0.0838**	**0.0066**
4.500	0.0200	0.0018	0.0036	**0.9079**	**0.0853**	**0.0068**
4.583	0.0200	0.0018	0.0036	**0.9063**	**0.0868**	**0.0069**
4.667	0.0200	0.0018	0.0037	**0.9046**	**0.0883**	**0.0071**
4.750	0.0200	0.0018	0.0037	**0.9030**	**0.0897**	**0.0073**
4.833	0.0200	0.0019	0.0037	**0.9014**	**0.0912**	**0.0074**
4.917	0.0200	0.0019	0.0038	**0.8997**	**0.0927**	**0.0076**
5.000	0.0200	0.0019	0.0038	**0.8981**	**0.0942**	**0.0078**

(b)

Time (t)	${}_t p^{00}_{50}$	${}_t p^{01}_{50}$	${}_t p^{02}_{50}$	Discount Factor	Death Benefit EPV	Expenses	Premium Annuity EPV
0.000	**1.0000**	**0.0000**	**0.0000**	1.0000	0.00	505.00	0.0833
0.083	**0.9982**	**0.0017**	**0.0001**	0.9950	47.78	4.97	0.0828
0.167	**0.9965**	**0.0033**	**0.0002**	0.9901	48.00	4.95	0.0822
0.250	**0.9947**	**0.0050**	**0.0003**	0.9851	48.21	4.92	0.0817
0.333	**0.9930**	**0.0066**	**0.0004**	0.9802	48.43	4.90	0.0811
0.417	**0.9912**	**0.0083**	**0.0005**	0.9754	48.65	4.87	0.0806
0.500	**0.9895**	**0.0100**	**0.0006**	0.9705	48.87	4.85	0.0800
0.583	**0.9877**	**0.0116**	**0.0007**	0.9657	49.09	4.83	0.0795
0.667	**0.9860**	**0.0132**	**0.0008**	0.9609	49.31	4.80	0.0789
0.750	**0.9842**	**0.0149**	**0.0009**	0.9561	49.54	4.78	0.0784
0.833	**0.9825**	**0.0165**	**0.0010**	0.9513	49.76	4.75	0.0779

Time (t)	${}_{t}p_{50}^{00}$	${}_{t}p_{50}^{01}$	${}_{t}p_{50}^{02}$	Discount Factor	Death Benefit EPV	Expenses	Premium Annuity EPV
0.917	**0.9807**	**0.0182**	**0.0011**	0.9466	49.99	4.73	0.0774
1.000	**0.9790**	**0.0198**	**0.0012**	0.9419	50.21	4.70	0.0768
1.083	**0.9773**	**0.0214**	**0.0013**	0.9372	50.44	4.68	0.0763
1.167	**0.9755**	**0.0230**	**0.0014**	0.9326	50.67	4.66	0.0758
1.250	**0.9738**	**0.0247**	**0.0015**	0.9279	50.91	4.63	0.0753
1.333	**0.9721**	**0.0263**	**0.0017**	0.9233	51.14	4.61	0.0748
1.417	**0.9703**	**0.0279**	**0.0018**	0.9187	51.38	4.59	0.0743
1.500	**0.9686**	**0.0295**	**0.0019**	0.9141	51.61	4.56	0.0738
1.583	**0.9669**	**0.0311**	**0.0020**	0.9096	51.85	4.54	0.0733
1.667	**0.9652**	**0.0327**	**0.0021**	0.9051	52.09	4.52	0.0728
1.750	**0.9635**	**0.0343**	**0.0022**	0.9006	52.33	4.49	0.0723
1.833	**0.9617**	**0.0359**	**0.0023**	0.8961	52.57	4.47	0.0718
1.917	**0.9600**	**0.0375**	**0.0025**	0.8916	52.81	4.45	0.0713
2.000	**0.9583**	**0.0391**	**0.0026**	0.8872	53.06	4.42	0.0709
2.083	**0.9566**	**0.0407**	**0.0027**	0.8828	53.31	4.40	0.0704
2.167	**0.9549**	**0.0423**	**0.0028**	0.8784	53.55	4.38	0.0699
2.250	**0.9532**	**0.0439**	**0.0029**	0.8740	53.80	4.36	0.0694
2.333	**0.9515**	**0.0454**	**0.0031**	0.8697	54.05	4.33	0.0690
2.417	**0.9498**	**0.0470**	**0.0032**	0.8653	54.31	4.31	0.0685
2.500	**0.9481**	**0.0486**	**0.0033**	0.8610	54.56	4.29	0.0680
2.583	**0.9464**	**0.0501**	**0.0034**	0.8567	54.82	4.27	0.0676
2.667	**0.9447**	**0.0517**	**0.0036**	0.8525	55.07	4.25	0.0671
2.750	**0.9430**	**0.0533**	**0.0037**	0.8482	55.33	4.23	0.0667
2.833	**0.9413**	**0.0548**	**0.0038**	0.8440	55.59	4.20	0.0662
2.917	**0.9396**	**0.0564**	**0.0040**	0.8398	55.85	4.18	0.0658
3.000	**0.9380**	**0.0579**	**0.0041**	0.8356	56.12	4.16	0.0653
3.083	**0.9363**	**0.0595**	**0.0042**	0.8315	56.38	4.14	0.0649
3.167	**0.9346**	**0.0610**	**0.0044**	0.8274	56.65	4.12	0.0644
3.250	**0.9329**	**0.0626**	**0.0045**	0.8232	56.92	4.10	0.0640
3.333	**0.9312**	**0.0641**	**0.0047**	0.8191	57.19	4.08	0.0636
3.417	**0.9296**	**0.0656**	**0.0048**	0.8151	57.46	4.06	0.0631
3.500	**0.9279**	**0.0672**	**0.0049**	0.8110	57.73	4.04	0.0627
3.583	**0.9262**	**0.0687**	**0.0051**	0.8070	58.00	4.01	0.0623
3.667	**0.9245**	**0.0702**	**0.0052**	0.8030	58.28	3.99	0.0619
3.750	**0.9229**	**0.0718**	**0.0054**	0.7990	58.56	3.97	0.0614
3.833	**0.9212**	**0.0733**	**0.0055**	0.7950	58.83	3.95	0.0610
3.917	**0.9195**	**0.0748**	**0.0057**	0.7910	59.12	3.93	0.0606
4.000	**0.9179**	**0.0763**	**0.0058**	0.7871	59.40	3.91	0.0602
4.083	**0.9162**	**0.0778**	**0.0060**	0.7832	59.68	3.89	0.0598
4.167	**0.9146**	**0.0793**	**0.0061**	0.7793	59.97	3.87	0.0594

Time (t)	${}_tp_{50}^{00}$	${}_tp_{50}^{01}$	${}_tp_{50}^{02}$	Discount Factor	Death Benefit EPV	Expenses	Premium Annuity EPV
4.250	**0.9129**	**0.0808**	**0.0063**	0.7754	60.25	3.85	0.0590
4.333	**0.9112**	**0.0823**	**0.0064**	0.7716	60.54	3.83	0.0586
4.417	**0.9096**	**0.0838**	**0.0066**	0.7677	60.83	3.81	0.0582
4.500	**0.9079**	**0.0853**	**0.0068**	0.7639	61.12	3.79	0.0578
4.583	**0.9063**	**0.0868**	**0.0069**	0.7601	61.42	3.77	0.0574
4.667	**0.9046**	**0.0883**	**0.0071**	0.7563	61.71	3.75	0.0570
4.750	**0.9030**	**0.0897**	**0.0073**	0.7525	62.01	3.74	0.0566
4.833	**0.9014**	**0.0912**	**0.0074**	0.7488	62.31	3.72	0.0562
4.917	**0.8997**	**0.0927**	**0.0076**	0.7451	62.61	3.70	0.0559
5.000	**0.8981**	**0.0942**	**0.0078**	0.7414	62.91		

EPV = 3294.96 (sum of Death Benefit EPV column)

(c) Premium
= (3294.96 + (sum of Expenses column) / (sum of Premium Annuity EPV column) / 12
= (3294.96 + 759.08) / 4.12 / 12 = 81.93

(d)

(1) A greater force of transition to the Disabled state will reduce the chances of being in the Healthy state and hence the expected number of premium payments. So the premium will need to be higher to compensate for the fewer expected number of premium payments.

(2) The force of transition to the Dead state is higher from the Disabled state than from the Healthy state, hence there will now be a larger probability of death, increasing the EPV (Benefits) and hence the gross premium.

Question 58

This question is an Excel question. Please see the ALTAM Sample Excel Solution file for details. The content of the Excel file for this question is provided below for reference.

(a)

Time (t)	Reserve at time t-1	Premium	Expenses	Interest	Expected cost of Claims	Expected Reserve at t	Profit Vector (a) (i)	Profit Signature (a) (ii)
0			300.0				**-300.0**	**-300.0**
1		3000	250.0	165.0	1041.3	940.1	**933.6**	**933.6**
2	1000	3000	252.5	224.9	1167.0	1408.4	**1397.0**	**1313.3**
3	1500	3000	255.1	254.7	1308.1	1875.1	**1316.4**	**1162.0**
4	2000	3000	257.7	284.5	1466.4	2340.2	**1220.3**	**1009.9**
5	2500	3000	260.4	314.4	1644.0	2803.1	**1106.8**	**857.4**
6	3000	3000	263.1	344.2	1843.3	3263.7	**974.0**	**705.0**
7	3500	3000	266.0	374.0	2066.8	3721.5	**819.8**	**553.3**
8	4000	3000	268.9	403.9	2317.5	4175.9	**641.6**	**402.9**
9	4500	3000	271.8	433.7	2598.4	4626.6	**436.9**	**254.6**
10	5000	3000	274.9	463.5	2913.2	5072.8	**202.6**	**109.3**
11	5500	3000	278.0	493.3	3265.8	5804.0	**-354.6**	**-176.3**
12	6000	3000	281.2	523.1	3660.7	5780.4	**-199.1**	**-95.8**
13	6000	3000	284.5	522.9	4102.5	5753.8	**-618.0**	**-286.4**
14	6000	3000	287.9	522.7	4596.8	5724.2	**-1086.1**	**-482.7**
15	6000	3000	291.3	522.5	5149.3	5691.0	**-1609.1**	**-682.3**
16	6000	3000	294.8	522.3	5766.5	5654.0	**-2193.0**	**-882.0**
17	6000	3000	298.5	522.1	6455.4	5612.7	**-2844.4**	**-1078.1**
18	6000	3000	302.2	521.9	7223.7	5566.6	**-3570.6**	**-1265.9**
19	6000	3000	306.0	521.6	8079.8	5515.2	**-4379.3**	**-1440.5**
20	6000	3000	309.9	521.4	9032.6	0.0	**179.0**	**54.1**

(b) Present value of Profit Signature column at 12%:
$= -300.0 + 933.6\times1.12^{-1} + 1313.3\times1.12^{-2} \ldots = 3441.59$

(c) Profit margin = 3441.59 / (Sumproduct(Premium, ${}_{t-1}p_{70}^{(t)}$,PV factors @ 12%) × 1.12
= 19.24%

(d) It is not advisable to have negative emerging profit in the later years of the contract. That requires the insurer to acquire additional cash from elsewhere, which may not be feasible.

(e) (i)

Time (t)	Reserve at time t-1	Premium	Expenses	Interest	Expected cost of Claims	Expected Reserve at t	Profit Vector (a) (i)	Profit Signature (a) (ii)
0			300.0				**-300.0**	**-300.0**
1	0	3000	250.0	165.0	1041.3	0.0	**1873.7**	**1873.7**
2	0	3000	252.5	164.9	1167.0	0.0	**1745.3**	**1640.8**
3	0	3000	255.1	164.7	1308.1	1407.1	**194.4**	**171.6**
4	1501	3000	257.7	254.6	1466.4	3031.4	**0.0**	**0.0**
5	3238	3000	260.4	358.7	1644.0	4692.7	**0.0**	**0.0**
6	5022	3000	263.1	465.5	1843.3	6381.3	**0.0**	**0.0**
7	6843	3000	266.0	574.6	2066.8	8085.1	**0.0**	**0.0**
8	8690	3000	268.9	685.3	2317.5	9789.2	**0.0**	**0.0**
9	10549	3000	271.8	796.6	2598.4	11475.3	**0.0**	**0.0**
10	12402	3000	274.9	907.6	2913.2	13121.0	**0.0**	**0.0**
11	14226	3000	278.0	1016.9	3265.8	14699.0	**0.0**	**0.0**
12	15195	3000	281.2	1074.8	3660.7	15328.3	**0.0**	**0.0**
13	15911	3000	284.5	1117.6	4102.5	15641.2	**0.0**	**0.0**
14	16310	3000	287.9	1141.4	4596.8	15567.1	**0.0**	**0.0**
15	16317	3000	291.3	1141.6	5149.3	15018.1	**0.0**	**0.0**
16	15833	3000	294.8	1112.3	5766.5	13884.5	**0.0**	**0.0**
17	14734	3000	298.5	1046.1	6455.4	12026.4	**0.0**	**0.0**
18	12856	3000	302.2	933.3	7223.7	9263.7	**0.0**	**0.0**
19	9985	3000	306.0	760.7	8079.8	5360.0	**0.0**	**0.0**
20	5831	3000	309.9	511.3	9032.6	0.0	**0.0**	**0.0**

(ii) NPV $= -300.0 + 1873.7 \times 1.12^{-1} + 1640.8 \times 1.12^{-2} + 171.6 \times 1.12^{-3} = 2803.113$

Question 59

This question is an Excel question. Please see the ALTAM Sample Excel Solution file for details. The content of the Excel file for this question is provided below for reference.

(a) The account value (AV) at age 90 is 84726.91.

Age	Year	Mort Rate	DB	Premiums	Expenses	i_q	i_c	Cor Fact	AV
60	1	0.003398	50,000	1,600	345	3%	5%	1.50	1,114.54
61	2	0.003792	50,000	1,600	153	3%	5%	1.48	2,469.14
62	3	0.004234	50,000	1,600	153	3%	5%	1.46	3,873.03
63	4	0.004730	50,000	1,600	153	3%	5%	1.44	5,327.55
64	5	0.005288	50,000	1,600	153	3%	5%	1.42	6,834.04
65	6	0.005915	50,000	1,600	153	3%	5%	1.40	8,394.04
66	7	0.006619	50,000	1,600	153	3%	5%	1.38	10,009.29
67	8	0.007409	50,000	1,600	153	3%	5%	1.36	11,681.81
68	9	0.008297	50,000	1,600	153	3%	5%	1.34	13,413.91
69	10	0.009294	50,000	1,600	153	3%	5%	1.32	15,208.40
70	11	0.010413	50,000	1,600	25	3%	5%	1.30	17,204.81
71	12	0.011670	50,000	1,600	25	3%	5%	1.28	19,280.25
72	13	0.013081	50,000	1,600	25	3%	5%	1.26	21,441.01
73	14	0.014664	50,000	1,600	25	3%	5%	1.24	23,694.94
74	15	0.016440	50,000	1,600	25	3%	5%	1.22	26,051.82
75	16	0.018433	50,000	1,600	25	3%	5%	1.20	28,523.89
76	17	0.020668	50,000	1,600	25	3%	5%	1.18	31,126.65
77	18	0.023175	50,000	1,600	25	3%	5%	1.16	33,879.73
78	19	0.025984	50,000	1,600	25	3%	5%	1.14	36,808.14
79	20	0.029132	50,000	1,600	25	3%	5%	1.12	39,943.93
80	21	0.032658	50,000	1,600	25	3%	5%	1.10	43,328.34
81	22	0.036607	50,000	1,600	25	3%	5%	1.08	46,980.20
82	23	0.041025	50,000	1,600	25	3%	5%	1.06	50,829.91
83	24	0.045968	50,000	1,600	25	3%	5%	1.04	54,901.66
84	25	0.051493	50,000	1,600	25	3%	5%	1.02	59,225.88
85	26	0.057665	50,000	1,600	25	3%	5%	1.00	63,840.92
86	27	0.064554	50,000	1,600	25	3%	5%	1.00	68,686.72
87	28	0.072237	50,000	1,600	25	3%	5%	1.00	73,774.80
88	29	0.080798	50,000	1,600	25	3%	5%	1.00	79,117.29
89	30	0.090326	50,000	1,600	25	3%	5%	1.00	84,726.91

(b)

- DB at end of 15th year = 50000.00
- DB at end of 20th year = 50000.00
- DB at end of 25th year = 60410.40

(c)

- CV at end of 1st year = 0.00
- CV at end of 2nd year = 669.14

(d) The main purpose of the surrender charge is to ensure that the insurer receives enough to pay the acquisition charges if the policy were to lapse.

(e) The annual premium is 957.94.

(f) The corridor factors do not affect the premium. This can be seen by noting that the AV based on a death benefit using the corridor factors is always larger than without the corridor factors. This is reasonable because with the lower premium, the cash value times the corridor factor is always less than 50,000.

(g) The year of termination is 27.

(h) A no lapse guarantee guarantees that if the policyholder pays a certain minimum premium each year, then the policy will stay in force even if the account value goes negative. The guarantee could apply if the expense charges and/or the mortality charges increase sufficiently to make the minimum premium insufficient.

Question 60

This question is an Excel question. Please see the ALTAM Sample Excel Solution file for details. The content of the Excel file for this question is provided below for reference.

a) The final salary is (3.698 + 3.643 + 3.589)/(3 × 3.186) × 100,000 = 114,354

b) See the table below:

Retirement Age (y)	Annuity Factor	Final Avg Salary	Discount Factor	Probability	Actuarial Liability
60	14.4414	106,152	0.78353	0.26675	136,172
60.5	14.3149	106,947	0.76464	0.05910	29,405
61.5	14.0559	108,548	0.72823	0.05324	25,138
62.5	13.7889	110,175	0.69355	0.04793	21,462
63.5	13.5139	111,828	0.66053	0.04313	18,298
64.5	13.2312	113,507	0.62907	0.03879	15,576
65	13.0870	114,354	0.61391	0.36765	143,556
				ANSWER:	389,607

c) The normal contribution is 15,584, which is simply 1/25 of the actuarial liability.

d) The purpose is to ensure that employees receive a fair result for the amount that they have contributed compared to other employees. It should ideally be calculated to make the actuarial value of their benefits at the date of retirement approximately equal to the present value of the benefits if they retire at the normal retirement age.

e) See the table below.

f) See the table below:

		Replacement Rate	
Age	**Reduction**	**w/ reduction**	**w/o reduction**
60	29.0%	35.7%	50.3%
60.5	26.6%	37.5%	51.1%
61.5	21.5%	41.4%	52.8%
62.5	16.0%	45.7%	54.4%
63.5	10.0%	50.5%	56.1%
64.5	3.5%	55.8%	57.8%
65	0.0%	58.6%	58.6%

Question 61

(a) Fund value at time 2 is $\left(\frac{S2}{S0}\right)10{,}000(0.97^2) = 9409\left(\frac{S2}{S0}\right)$
GMDB during year 2 is $10{,}000e^{2(0.06)} = 11{,}274.97$.
Those are equal if $\left(\frac{S_2}{S_0}\right) = \frac{11{,}274.97}{9409} = 1.1983$. Anything higher, and the death benefit will be the fund value.

(b) The Death Benefit for death in year 1 is

$$\max(10{,}000e^{0.06}, F_1) = \max\left(10{,}000e^{0.06}, 0.97(10{,}000)\frac{S1}{S0}\right)$$
$$= 9{,}700\max\left(\frac{e^{0.06}}{0.97} - \frac{S_1}{S_0}, 0\right)$$

which is 9,700 times the payoff of a put option with strike price $\frac{e^{0.06}}{0.97} = 1.09468$.

(i)

$$d_1(0,1) = \frac{\left[ln\left(\frac{(1-0.03)^1}{e^{0.06(1)}}\right) + \left(0.04 + \frac{0.30^2}{2}\right)(1)\right]}{[(0.30)1^{0.5}]} = -0.018197$$

(ii)

$$d_2(0,1) = d_1(0,1) - \left[(0.30)1^{0.5}\right] = -0.318197$$
$$v(0,1) = 9{,}700[1.09468\, e^{-0.04}\, NORM.S.DIST(0.318197{,}1)$$
$$- NORM.S.DIST(0.018197{,}1)] = 1{,}454.13$$

Contribution to hedge portfolio = q_{85} $(1{,}454.13) = 0.057665(1{,}454.13) = 83.85$

(iii)

$$d_1(0,2) = \frac{\left[ln\left(\frac{(1-0.03)^2}{e^{0.06(2)}}\right) + \left(0.04 + \frac{0.30^2}{2}\right)(2)\right]}{[(0.30)2^{0.5}]} = -0.025735$$
$$d_2(0,2) = d_1(0,2) - \left[(0.30)2^{0.5}\right] = -0.449999$$
$$v(0,2) = 10{,}000(0.97^2)\left[1.09468^2\, e^{-0.04(2)}\, NORM.S.DIST(0.44999{,}1)\right.$$
$$\left. - NORM.S.DIST(0.025735{,}1)\right] = 2{,}210.27$$

Contribution to hedge portfolio = ${}_{1|}q_{85}$ $(2{,}210.27) = 0.060831(2{,}210.27) = 134.45$
Hedge portfolio $= 83.85 + 134.45 = 218.30$

(iv) (See solution to Example 17.6 for how this formula arises)

$$218.30 = 10{,}000\, c\,(1 + (1-0.03)p_{85}) = 10{,}000(1 + 0.97(0.942335))\, c$$
$$= 19{,}140.65\, c$$

$c = 0.0114$

(v) The cost would not change. Chapter 17 of AMLCR uses the convention that $S_0 = 1$, but S_0 is an index of stock prices. (As noted on page 632, “We interpret the stock price process as an index for the fund assets; as an index. We can arbitrarily set S_0 to any convenient value.)

(c)

(i)

$$d_1 = \left[ln\left(\frac{(1-0.03)^1}{e^{0.06(1)}}\right) + \left(0.04 + \frac{0.30^2}{2}\right)(1)\right] / \left[(0.30)1^{0.5}\right] = -0.018197$$

Exactly the same as before. The exact value of the assets on the calculation of the hedge portfolio does not appear, because it is irrelevant. Asset values follow the same lognormal price process from that point, and the GMDBs (the strike prices) are the same multiples of the starting assets)

So, the ratio of the d_1's is 1.000000000…

(ii) Similarly, d_2 is the same as before.
The assets at the end of year 1 are $10{,}000(2)(0.97) = 19{,}400$
The Reset Death Benefit for death in year 1 is

$$max(10{,}000e^{0.06}, F_2) = max\left(19{,}400e^{0.06}, 0.97(19{,}400)\frac{S_2}{S_1}\right)$$
$$= 18{,}818\, max\left(\frac{e^{0.06}}{0.97} - \frac{S_2}{S_1}, 0\right)$$

which is 18,818 times the payoff of a 1-year put option at time 1 with strike price $\frac{e^{0.06}}{0.97} = 1.09468$

$$v(1,1) = 18{,}818[1.09468\, e^{-0.04}\, NORM.S.DIST(0.318197,1) - NORM.S.DIST(0.018197,1)] = 2821.02$$

Contribution to hedge portfolio is $q_{86}(2{,}821.02) = 0.064555(2{,}821.02) = 182.11$

$$v(1,2) = 18{,}818(0.97)[1.09468^2\, e^{-0.04(2)}\, NORM.S.DIST(0.44999,1) - NORM.S.DIST(0.025735,1)] = 4{,}287.93$$

Contribution to hedge portfolio is $= {}_{1|}q_{86}\,(4{,}287.93) = 0.067574(4{,}287.93) = 289.76$
Hedge portfolio $= 182.11 + 289.76 = 471.87$

(d) Your colleague is wrong. Expected values are additive, whether or not the underlying events are independent.