Interaction of Market and Credit Risk: An Analysis of Inter-Risk Correlation and Risk Aggregation

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Abstract

In this paper we investigate the interaction between a credit portfolio and another risk type, which can be thought of as market risk. Combining Merton-like factor models for credit risk with linear factor models for market risk, we analytically calculate their inter-risk correlation and show how inter-risk correlation bounds can be derived. Moreover, we elaborate how our model naturally leads to a Gaussian copula approach for describing dependence between both risk types. In particular, we suggest estimators for the correlation parameter of the Gaussian copula that can be used for general credit portfolios. Finally, we use our findings to calculate aggregated risk capital of a sample portfolio both by numerical and analytical techniques.

1 Introduction

A core element of modern risk management and control is analyzing the *capital adequacy* of a financial institution, which is concerned with the assessment of the firm's required capital to cover the risks it takes. To this end, financial firms seek to quantify their overall risk exposure by aggregating all individual risks associated with different risk types or business units, and to compare this figure with a so-called *risk taking capacity*, defined as the total amount of capital as a buffer against potential losses.

Until now no standard procedure for risk aggregation has emerged, but, according to an industry survey of The Joint Forum [2], a widespread approach in the banking industry

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is *aggregation across risk types* where the marginal loss distributions of all relevant risk types are independently modelled from their dependence structure. This approach splits up into three steps:

- First, assign every individual risk position to a certain risk type.
- Second, calculate an aggregated measure for every risk type encompassing all business units by using separate, risk-type specific techniques and methodologies.
- Third, integrate all pre-aggregated risk figures of different risk types to obtain the overall capital number, henceforth called *aggregated risk capital.*

The easiest solution for the last step is simply to add up all pre-aggregated risk figures. This, however, is only a rough estimate of the bank-wide total risk exposure. Moreover, banks usually try to reduce their overall risk by accounting for diversification between different risk types because it allows them either to reduce their capital buffer (and thus expensive equity capital) or to increase their business volume (and thus to generate additional earnings). As a consequence thereof, return on equity and eventually shareholder value increase. Hence, advanced approaches for risk aggregation begin with an analysis of the dependence structure between different risk types.

In this paper, we combine a Merton-like factor model for credit risk with a linear factor model for another risk type—henceforth referred to as market risk—and investigate their correlation and the resulting aggregate risk. Both models are driven by a set of (macroeconomic) factors $Y = (Y_1, \dots, Y_K)$ where the factor weights are allowed to be zero so that a risk type may only depend on a subset of Y.

As an important measure of association, we start with an in-depth analysis of linear correlation between both risk types (henceforth referred to as *inter-risk correlation*). Our approach allows us to derive closed-form expressions for inter-risk correlation in the case of normally distributed and heavy-tailed risk factors, providing valuable insight into inter-risk dependence of a credit risk portfolio in general. In particular, we give upper bounds for inter-risk correlation, which only depend on typical credit portfolio characteristics such as its asset correlation or rating structure.

A very natural integration technique, especially in the context of aggregation across risk types, is based on copulas, see, e.g., Dimakos & Aas [5], Rosenberg & Schuermann [10]), Ward & Lee [11], or Böcker & Spielberg [4]. As a result of Sklar's theorem, copulas allow for a separate modelling of marginal distribution functions (second step above) on one hand and their dependence structure (third step above) on the other hand. However, the choice and parametrization of a copula is usually not straightforward, especially in the context of risk aggregation where reliable data are often difficult to obtain. We show that for large homogenous portfolios our model quite naturally leads to a Gaussian coupling model between both risk types and provides a simple estimator for the copula correlation parameter, which can be used as an approximation also in the case of more general credit portfolios.

Finally, we perform a simulation study where we apply our findings to a test portfolio, for which aggregated risk capital is calculated by means of the copula technique as well as the well-known *square-root-formula* approach. Though mathematically justified only in the case of elliptically distributed risk types (with the multivariate normal or t distributions as prominent examples), this approach is very often used as a first approximation because total aggregated capital can then be calculated explicitly without (time-)expensive simulations, see, e.g., The IFRI/CRO Forum [7], The Joint Forum [2] or Rosenberg & Schuermann [10]. If $\mathrm{EC}^T = (\mathrm{EC}_1, \ldots, \mathrm{EC}_m)$ is the vector of pre-aggregated risk figures (e.g., economic capital EC_i for risk-types $i = 1, \ldots, m$ as defined in Section 5), and R the inter-risk correlation matrix, then total aggregated risk EC_{tot} is estimated via

$$\mathrm{EC}_{tot} = \sqrt{\mathrm{EC}^T R\, \mathrm{EC}}\,. \tag{1.1}$$

Hence, a typical problem of risk aggregation is the estimation of the inter-risk correlation matrix R. While we observe that the square-root-formula seriously underestimates aggregated risk capital in the case of a Student t copula between market and credit risk, it seems to be a quite reasonable approximation if a Gaussian dependence structure is assumed.

2 Preliminaries: Modelling Credit and Market Risk

2.1 Factor Models for Credit Risk

To describe credit portfolio loss, we choose a classical structural model as it can be found for example in Bluhm, Overbeck and Wagner [3]. Within these models, a borrower's credit quality (and so his default behaviour) is driven by its *asset-value process*, or, more generally and in the case of unlisted customers, by a so-called "ability-to-pay" process. Consider a portfolio of n loans. Then, default of an individual obligor i is described by a Bernoulli random variable $L_i, i = 1, \ldots, n$, with $\mathbb{P}(L_i = 1) = p_i$ and $\mathbb{P}(L_i = 0) = 1 - p_i$ where p_i is the obligor's *probability of default* within time period $[0, T]$ for $T > 0$. Following Merton's idea, counterparty i defaults if its asset value log-return A_i falls below some threshold D_i, sometimes referred to as *default point*, i.e.

$$L_i = \mathbb{1}_{\{A_i < D_i\}}\,, \qquad i = 1, \ldots, n\,.$$

If we denote the exposure at default net recovery of an individual obligor by e_i, portfolio loss is finally given by

$$L^{(n)} = \sum_{i=1}^{n} e_i \, L_i \,. \tag{2.1}$$

For a credit portfolio of n obligors, credit portfolio loss $L^{(n)}$ at time horizon T is driven by n realizations of the asset values A_i, which usually are assumed to depend on factors $(Y_1, \ldots, Y_K)$. The following factor model is widely spread in financial firms and similar versions are implemented in various software packages for credit risk.

Definition 2.1. [Normal factor model for credit risk] *Let $Y = (Y_1, \ldots, Y_K)$ be a K-dimensional random vector of (macroeconomic) factors with multivariate standard normal distribution. Then, in the normal factor model, each of the standardized asset value log-returns A_i, $i = 1, \ldots, n$, depends linearly on Y and a standard normally distributed idiosyncratic factor (or noise term) ε_i, independent of all Y_k, i.e.*

$$A_i = \sum_{k=1}^{K} \beta_{ik} Y_k + \sqrt{1 - \sum_{k=1}^{K} \beta_{ik}^2} \; \varepsilon_i \,, \qquad i = 1, \ldots, n \,, \tag{2.2}$$

with factor loadings β_{ik} satisfying $R_i^2 := \sum_{k=1}^{K} \beta_{ik}^2 \in [0,1]$ describing the variance of A_i that can be explained by the systematic factors Y.

For later usage we recall some properties of the normal factor model as can be found, e.g., in Bluhm et al. [3] or McNeil, Frey and Embrechts [9], Chapter 8.

Remark 2.2. (a) The log-returns $A_1, \ldots, A_n$ are standard normally distributed and dependent with correlations

$$\rho_{ij} := \operatorname{corr}(A_i, A_j) = \sum_{k=1}^{K} \beta_{ik} \beta_{jk} \,, \qquad i, j = 1, \ldots, n \,, \tag{2.3}$$

the so-called *asset correlations* between A_i and A_j.
(b) The default point D_i of every obligor is related to its default probability p_i by

$$D_i = \Phi^{-1}(p_i) \,, \qquad i = 1, \ldots, n \,, \tag{2.4}$$

where Φ is the standard normal distribution function.
(c) The joint default probability of two obligors is given by

$$p_{ij} := \mathbb{P}(L_i = 1, L_j = 1) = \mathbb{P}(A_i \le D_i, A_j \le D_j) = \begin{cases} \Phi_{\rho_{ij}}(D_i, D_j) \,, & i \neq j \,, \\ p_i \,, & i = j \,, \end{cases} \tag{2.5}$$

where $\Phi_{\rho_{ij}}$ denotes the bivariate normal distribution function with standard marginals and correlation ρ_{ij} given by (2.3). Moreover, the *default correlation* between two different obligors is given by

$$\operatorname{corr}(L_i, L_j) = \frac{p_{ij} - p_i\, p_j}{\sqrt{p_i(1-p_i)\, p_j(1-p_j)}}\,, \qquad i,j = 1,\dots,n\,. \tag{2.6}$$

(d) Conditional on a realization $y = (y_1, \dots, y_K)$ of the factors Y, defaults of different obligors are independent. Moreover, the *conditional default probability* is given by

$$\begin{aligned} p_i(y) &= \mathbb{P}(L_i = 1 \,|\, Y = y) \\ &= \mathbb{P}\left(\sum_{k=1}^{K} \beta_{ik}\, y_k + \sqrt{1 - \sum_{k=1}^{K} \beta_{ik}^2}\; \varepsilon_i \le D_i \right) \\ &= \Phi\left(\frac{D_i - \sum_{k=1}^{K} \beta_{ik}\, y_k}{\sqrt{1 - \sum_{k=1}^{K} \beta_{ik}^2}} \right). \end{aligned}$$

□

A strong assumption of the model above is the multivariate normal distribution of the factor variables $Y = (Y_1, \dots, Y_K)$, and thus of the asset value log-returns A_i. It is well known that the normal distribution has very light tails and therefore may seriously underestimate large fluctuations of the (macroeconomic) factors, eventually leading to model risk of the normal factor model for credit risk.

A generalization allowing for heavier tails as well as a stronger dependence between different counterparties is the class of normal variance mixture distributions, where the covariance structure of the A_i is disturbed by means of a positive mixing variable W_L (see, e.g., McNeil et al. [9], Section 3.2). A particularly interesting model is the following one (confer also Kostadinov [8]):

Definition 2.3. [Shock model for credit risk] *Let $Y = (Y_1, \dots, Y_K)$ be a K-dimensional random vector of (macroeconomic) factors with multivariate standard normal distribution. Then, in the shock model, each of the standardized asset value log-returns $\widehat{A}_i$, $i = 1, \dots, n$, can be written as*

$$\widehat{A}_i = W_L \cdot \sum_{k=1}^{K} \beta_{ik} Y_k + W_L \cdot \sqrt{1 - \sum_{k=1}^{K} \beta_{ik}^2}\; \varepsilon_i\,, \qquad i = 1, \dots, n\,, \tag{2.7}$$

where $W_L = \sqrt{\nu_L / S_{\nu_L}}$ and S_{ν_L} is a $\chi^2_{\nu_L}$ distributed random variable with ν_L degrees of freedom, independent of Y and the idiosyncratic factor ε_i.

The mixing variable W_L can be interpreted as a "global shock" driving the variance of all factors. Such an overarching shock may occur from political distress, severe economic recession or some natural disaster.

We conclude this section with some general remarks about the shock model for credit risk (see again Bluhm et al. [3] and McNeil et al. [9] as well as references therein).

Remark 2.4. (a) In general, let $X = (X_1, \ldots, X_n)$ be a standardized multinormal vector with covariance matrix R and S_{ν_L} is a chi-square variable with ν_L degrees of freedom. Then $(X_1, \ldots, X_n)/\sqrt{S_{\nu_L}/\nu_L}$ has a multivariate t-distribution with correlation matrix R and ν_L degrees of freedom. Hence, from (2.2) and (2.7) it follows for the shock model for credit risk that the vector of standardized asset value log-returns $(\widehat{A}_1, \ldots, \widehat{A}_n)$ is t-distributed with ν_L degrees of freedom; in particular, it has the same asset correlation ρ_{ij} as the normal factor model given by equation (2.3).
(b) The default point $\widehat{D}_i$ of the shock model is linked to the obligor's default probability by

$$\widehat{D}_i = t_{\nu_L}^{-1}(p_i)\,, \qquad i = 1, \ldots, n\,, \tag{2.8}$$

where t_{ν_L} is the Student t distribution function with ν_L degrees of freedom.
(c) The joint default probability $\widehat{p}_{ij}$ for two obligors can be written as

$$\widehat{p}_{ij} = t_{\nu_L;\rho_{ij}}(\widehat{D}_i, \widehat{D}_j)\,, \qquad i \neq j\,, \tag{2.9}$$

where $t_{\nu_L;\rho_{ij}}$ denotes the standard bivariate Student t distribution function with correlation ρ_{ij} given by (2.3) and degree of freedom parameter ν_L. $\square$

2.2 Joint Factor Models for Credit and Market Risk

Market risk is related to a bank's potential loss associated with its trading activities. We assume that it is already pre-aggregated so that losses can be approximately described by a one-dimensional random variable Z (or $\widehat{Z}$, see below), which can be thought of as the bank-wide, aggregated profit and loss (P/L) distribution due to changes in some market variables, such as interest rates, foreign exchange rates, equity prices or the value of commodities.

Similarly as for credit risk, we explain fluctuations of the random variable Z by means of (macroeconomic) factors $(Y_1, \ldots, Y_K)$. We use the same macroeconomic factors for credit and market risk, where independence of risk from such a factor is indicated by a loading factor 0. If the pre-aggregated P/L can be described by a normal distribution, the following factor model is a sensible choice and can be used for risk aggregation. Even if this assumption does not hold exactly, it can be used as an important approximation for investigating inter-risk dependencies (we use the convention that losses correspond to positive values of Z).

Definition 2.5. [Normal factor model for market risk] *Let* $Y = (Y_1, \dots, Y_K)$ *be a random vector of (macroeconomic) factors with multivariate standard normal distribution. Then, the normal factor model for the pre-aggregated market risk P/L is given by*

$$Z = -\sigma \left(\sum_{k=1}^{K} \gamma_k Y_k + \sqrt{1 - \sum_{k=1}^{K} \gamma_k^2 \, \eta} \right), \tag{2.10}$$

with factor loadings γ_k *satisfying* $\sum_{k=1}^{K} \gamma_k^2 \in [0,1]$, *which is that part of the variance of* Z *which can be explained by the systematic factor* Y*. Furthermore,* η *is a standard normally distributed idiosyncratic factor, independent of* Y*, and* σ *is the standard deviation of* Z*.*

Clearly, for an actively managed market portfolio the idiosyncratic factor η is more important than for an unmanaged portfolio (e.g., an index of stocks). As a matter of fact, portfolio managers are paid owing to their skills to achieve the best possible portfolio performance that is independent of some macroeconomic indicators.

Note that both in Definition 2.1 of the credit factor model as well as above, the factor loadings β_{ik} and γ_k, respectively, are allowed to be zero. For instance, Y_k can be relevant for credit but not for market by setting $\gamma_k = 0$.

Definition 2.6. [Joint normal factor model for credit and market risk] *Let* $Y = (Y_1, \dots, Y_K)$ *be a random vector of (macroeconomic) factors with multivariate standard normal distribution. Let the credit portfolio loss* $L^{(n)}$ *be given by* (2.1), *and the asset value log-returns* A_i *for* $i = 1, \dots, n$ *be modeled by the normal factor model* (2.2)*. Let* Z *be the pre-aggregated market risk P/L modeled by the normal factor model* (2.10)*. When the idiosyncratic factors* ε_i *for* $i = 1, \dots, n$ *of the credit model are independent of* η*, then we call* $(L^{(n)}, Z)$ *the* joint normal factor model for credit and market risk*.*

In order to account for possible heavy tails in the market risk P/L, we again rely on the global shock approach already used for credit risk.

Definition 2.7. [Shock model for market risk] *Let* $Y = (Y_1, \dots, Y_K)$ *be a random vector of (macroeconomic) factors with multivariate standard normal distribution. Then the shock model for the pre-aggregated market risk P/L is given by*

$$\widehat{Z} = -\sigma \left(W_Z \cdot \sum_{k=1}^{K} \gamma_k Y_k + W_Z \cdot \sqrt{1 - \sum_{k=1}^{K} \gamma_k^2 \, \eta} \right), \tag{2.11}$$

where σ *is a scaling factor,* $W_Z = \sqrt{\nu_Z / S_{\nu_Z}}$*, and* S_{ν_Z} *is a* $\chi^2_{\nu_Z}$ *distributed random variable with* ν_Z *degrees of freedom, independent of* Y *and the idiosyncratic factor* η*.*

Definition 2.8. [Joint shock model for credit and market risk] *Let* $Y = (Y_1, \dots, Y_K)$ *be a random vector of (macroeconomic) factors with multivariate standard normal distribution. Let the credit portfolio loss be given by* (2.1)*, now denoted as* $\widehat{L}^{(n)}$*, and the asset value log-returns* $\widehat{A}_i$ *for* $i = 1, \dots, n$ *be modeled by the shock model* (2.7) *with shock variable* W_L*. Let* $\widehat{Z}$ *be the pre-aggregated market risk P/L modeled by the shock model* (2.11) *with shock variable* W_Z*.*
(1) (Independent shock model for credit and market risk). *If the credit model's idiosyncratic factors* ε_i *for* $i = 1, \dots, n$ *are independent of* η*, and furthermore* W_L *is independent from* W_Z*, then we call* $(\widehat{L}^{(n)}, \widehat{Z})$ *the* independent shock model for credit and market risk*.*
(2) (Common shock model for credit and market risk). *If the credit model's idiosyncratic factors* ε_i *for* $i = 1, \dots, n$ *are independent of* η*, and furthermore if we set*

$$W_L \equiv W_Z =: W,$$

then we call $(\widehat{L}^{(n)}, \widehat{Z})$ *the* common shock model for credit and market risk*.*

3 Inter-Risk Correlation

3.1 Normal Factor Model Approach

The proposed models shall now be used to investigate the dependence between credit risk $L^{(n)}$ and market risk Z, introduced by the factors Y. Let us start with the linear correlation, which is defined as

$$\operatorname{corr}(L^{(n)}, Z) = \frac{\operatorname{cov}(L^{(n)}, Z)}{\sqrt{\operatorname{var}(L^{(n)})}\sqrt{\operatorname{var}(Z)}}\,. \tag{3.1}$$

Although linear correlation only describes linear dependence between different random variables, it is a very popular and important concept in finance, frequently used both by practitioners and academics. Moreover, since we calculate expressions for linear correlation in closed form, we are able to analytically investigate the linear dependence structure between market and credit risk. Note also that the correlation completely describes the dependence in the joint normal factor model.

We begin with the joint normal factor model for credit and market risk. Here as well as for all subsequent results, all proofs are given in the appendix.

Theorem 3.1 (Inter-risk correlation for the normal factor model)**.** *Suppose that credit portfolio loss* $L^{(n)}$ *and market risk* Z *are described by the joint normal factor models of Definition 2.6. Then, linear correlation between* $L^{(n)}$ *and* Z *is given by*

$$\operatorname{corr}(L^{(n)}, Z) = \frac{\sum_{i=1}^{n} r_i\, e_i \exp\left(-\frac{1}{2} D_i^2\right)}{\sqrt{2\pi \operatorname{var}(L^{(n)})}}\,, \tag{3.2}$$

where D_i is the default point (2.4),

$$r_i := \operatorname{corr}(A_i, Z) = \sum_{k=1}^{K} \beta_{ik}\gamma_k\,, \qquad i = 1, \ldots, n,$$

and

$$\operatorname{var}(L^{(n)}) = \sum_{i,j=1}^{n} e_i\, e_j\, (p_{ij} - p_i\, p_j)$$

with joint default probability p_{ij} given by (2.5).

Note that r_i may become negative if (some) factor loadings β_{ik} and γ_k have different signs. Therefore, in principle, also negative inter-risk correlations can be achieved in our model. Moreover, in (3.2) the term $e_i\, e^{-D_i^2/2}$ can be interpreted as a kind of rating-adjusted exposure. For instance, a relatively low default probability of debtor i corresponds to a relatively small value of $e^{-D_i^2/2}$. As a consequence thereof, for two obligors with equal exposure size e_i, the one with the better rating has less impact on inter-risk correlation as the low-rated creditor.

The fact that $\operatorname{corr}(L^{(n)}, Z)$ linearly depends on the correlation r_i and thus on the factor loadings γ_k, implies the following Proposition.

Proposition 3.2 (Inter-risk correlation bound for the joint normal factor model)**.** *Suppose that credit portfolio loss $L^{(n)}$ is described by the normal factor model of Definition 2.1, and market risk Z by the normal factor model of Definition 2.5, however, with unknown factor loadings γ_k, $k = 1, \ldots, K$. Then, inter-risk correlation is bounded by*

$$|\operatorname{corr}(L^{(n)}, Z)| \;\; \leq \;\; \frac{\sum_{i=1}^{n} e_i\, R_i \exp\left(-\frac{1}{2} D_i^2\right)}{\sqrt{2\pi \operatorname{var}(L^{(n)})}} \tag{3.3}$$

with $R_i = \sqrt{\sum_{k=1}^{K} \beta_{ik}^2}$.

Note that (3.3) does not depend on any specific market risk parameter. Therefore, solely based on the parametrization of the normal credit factor model, a bound for inter-risk correlation can be derived. This bound then holds for all market risk portfolios described by Definition 2.5. Furthermore, as R_i^2 is that part of the variance of A_i which can be explained by the factor Y, it follows from (3.3) that the inter-risk correlation bound is affine linearly increasing with R_i. This is also intuitively clear because with increasing R_i^2, credit portfolio loss is more and more dominated by the systematic factor Y, which by construction drives the inter-risk dependence with market risk.

3.2 Shock Model Approach

We now investigate how the existence of global shocks affects inter-risk correlation. We consider both kinds of joint shock models for credit and market risk given by Definition 2.8 and calculate inter-risk correlation similarly to Theorem 3.1.

Theorem 3.3 (Inter-risk correlation for the joint shock model)**.** *Suppose that credit portfolio loss $\widehat{L}^{(n)}$ and market risk $\widehat{Z}$ are described by the joint shock factor model of Definition 2.8.*
(1) (Independent shock model, Definition 2.8 (1)). *If shocks in credit and market risk are driven by different independent shock variables W_L and W_Z with degrees of freedom $\nu_L > 0$ and $\nu_Z > 2$, respectively, linear correlation between $\widehat{L}^{(n)}$ and $\widehat{Z}$ is given by*

$$\operatorname{corr}(\widehat{L}^{(n)}, \widehat{Z}) = \sqrt{\frac{\nu_Z - 2}{2}} \, \frac{\Gamma\left(\frac{\nu_Z - 1}{2}\right)}{\Gamma\left(\frac{\nu_Z}{2}\right)} \, \frac{\sum_{i=1}^{n} e_i \, r_i \left(1 + \frac{\widehat{D}_i^2}{\nu_L}\right)^{-\frac{\nu_L}{2}}}{\sqrt{2\,\pi \operatorname{var}(\widehat{L}^{(n)})}} \,. \tag{3.4}$$

(2) (Common shock model, Definition 2.8 (2)). *If shocks in credit and market risk are driven by the same shock variable W with $\nu > 1$ degrees of freedom, linear correlation between $\widehat{L}^{(n)}$ and $\widehat{Z}$ is given by*

$$\operatorname{corr}(\widehat{L}^{(n)}, \widehat{Z}) = \sqrt{\frac{\nu - 2}{2}} \, \frac{\Gamma\left(\frac{\nu - 1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \, \frac{\sum_{i=1}^{n} e_i \, r_i \left(1 + \frac{\widehat{D}_i^2}{\nu}\right)^{\frac{1-\nu}{2}}}{\sqrt{2\,\pi \operatorname{var}(\widehat{L}^{(n)})}} \,. \tag{3.5}$$

In both cases,

$$r_i := \operatorname{corr}(\widehat{A}_i, \widehat{Z}) = \sum_{k=1}^{K} \beta_{ik} \gamma_k \,, \qquad i = 1, \ldots, n,$$

and

$$\operatorname{var}(\widehat{L}^{(n)}) = \sum_{i,j=1}^{n} e_i \, e_j \, (p_{ij} - p_i \, p_j) \,.$$

Furthermore, $\widehat{D}_i$ and $\widehat{p}_{ij}$ are given by (2.8) *and* (2.9)*, respectively, with degree of freedom ν_L for the independent shock model (1) and ν for the common shock model (2).*

Analogously to the normal factor model, inter-correlation bounds can be derived.

Proposition 3.4 (Inter-risk correlation bounds for the joint shock model)**.** *Suppose that credit portfolio loss $\widehat{L}^{(n)}$ is described by the shock model of Definition 2.3 and market risk $\widehat{Z}$ by the shock model of Definition 2.7, however, with unknown factor loadings γ_k, $k =$*

$1, \ldots, K$.
(1) *For the independent shock model, inter-risk correlation is bounded by*

$$|\mathrm{corr}(\widehat{L}^{(n)}, \widehat{Z})| \;\leq\; \sqrt{\frac{\nu_Z - 2}{2}}\, \frac{\Gamma\left(\frac{\nu_Z - 1}{2}\right)}{\Gamma\left(\frac{\nu_Z}{2}\right)}\, \frac{\sum_{i=1}^{n} e_i\, R_i \left(1 + \frac{\widehat{D}_i^2}{\nu_L}\right)^{-\frac{\nu_L}{2}}}{\sqrt{2\,\pi\, \mathrm{var}(\widehat{L}^{(n)})}}\,. \tag{3.6}$$

(2) *For the common shock model, inter-risk correlation is bounded by*

$$|\mathrm{corr}(\widehat{L}^{(n)}, \widehat{Z})| \;\leq\; \sqrt{\frac{\nu - 2}{2}}\, \frac{\Gamma\left(\frac{\nu - 1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\, \frac{\sum_{i=1}^{n} e_i\, R_i \left(1 + \frac{\widehat{D}_i^2}{\nu}\right)^{\frac{1-\nu}{2}}}{\sqrt{2\,\pi\, \mathrm{var}(\widehat{L}^{(n)})}}\,. \tag{3.7}$$

In both cases (3.6) *and* (3.7) *is* $R_i = \sqrt{\sum_{k=1}^{K} \beta_{ik}^2}$.

For practical purposes, very relevant is the situation where credit risk quantification is based on a normal factor model, whereas heavy tails are assumed for the market risk, which therefore shall be described by the shock model approach. This can be referred to as a *hybrid factor model*, which is a special case of the joint shock model of Definition 2.3 with $\nu_L \to \infty$. We formulate our results as a Corollary.

Corollary 3.5 (Inter-risk correlation for the hybrid factor model)**.** *Suppose that credit portfolio loss* $L^{(n)}$ *is described by the normal factor model of Definition 2.1, and market risk* $\widehat{Z}$ *by the shock model of Definition 2.7. Assume that the credit model's idiosyncratic factors* ε_i *for* $i = 1, \ldots, n$ *are independent of* η*, then we call* $(L^{(n)}, \widehat{Z})$ *the* hybrid factor model*.*
(1) *Inter-risk correlation is given by*

$$\mathrm{corr}(L, \widehat{Z}) = \sqrt{\frac{\nu_Z - 2}{2}}\, \frac{\Gamma\left(\frac{\nu_Z - 1}{2}\right)}{\Gamma\left(\frac{\nu_Z}{2}\right)}\, \mathrm{corr}(L, Z)$$

with $\mathrm{corr}(L, Z)$ *as in* (3.2)*.*
(2) *If the factor loadings* γ_k, $k = 1, \ldots, K$ *of market risk are unknown, the inter-risk correlation bound is given by*

$$|\mathrm{corr}(L, \widehat{Z})| \leq \sqrt{\frac{\nu_Z - 2}{2}}\, \frac{\Gamma\left(\frac{\nu_Z - 1}{2}\right)}{\Gamma\left(\frac{\nu_Z}{2}\right)}\, |\mathrm{corr}(L, Z)|$$

with $|\mathrm{corr}(L, Z)|$ *as in* (3.3)*.*

4 An Application to One-Factor Models

4.1 Joint One-Factor Models for Credit and Market Risk

Instructive examples regarding inter-risk correlation and its bounds can be obtained for one-factor models, and they are useful to explain general characteristics and systematic behaviour of inter-risk correlation. In the context of credit risk, one-factor models can quite naturally be obtained by considering the special case of a large homogenous portfolio (LHP).

Let us start with a homogenous portfolio for which we define that $e_i = e$, $p_i = p$, $\beta_{ik} = \beta_k$ for $i = 1, \ldots, n$, and $k = 1, \ldots, K$. Then, by setting $\rho := \sum_{k=1}^{K} \beta_k^2$ and

$$\widetilde{Y} := \left(\sum_{k=1}^{K} \beta_k Y_k \right) / \sqrt{\rho} \,, \tag{4.1}$$

expression (2.2) for the general factor model can be transformed into a one-factor model

$$A_i = \sqrt{\rho}\, \widetilde{Y} + \sqrt{1-\rho}\, \varepsilon_i \,, \tag{4.2}$$

where $\widetilde{Y}$ is standard normally distributed and independent of ε_i, and ρ is the *uniform asset correlation* within the credit portfolio. If we now additionally increase the number of counterparties in the portfolio by $n \to \infty$, then the relative portfolio loss satisfies[1]

$$\frac{L^{(n)}}{n\, e} \overset{\text{a.s.}}{\to} \Phi \left(\frac{D - \sqrt{\rho}\, \widetilde{Y}}{\sqrt{1-\rho}} \right) =: L \,, \qquad n \to \infty \,, \tag{4.3}$$

where $D = \Phi^{-1}(p)$ and $n\, e$ is the total exposure of the credit portfolio. Often L is used as an approximative loss variable for large and almost homogeneous portfolios. For later usage recall that the variance of L is given by $\mathrm{var}(L) = p_{12} - p^2$ with $p_{12} = \Phi_\rho(D, D)$. Similarly, in the case of the shock model, the LHP approximation reads

$$\frac{\widehat{L}^{(n)}}{n\, e} \overset{\text{a.s.}}{\to} \Phi \left(\frac{\widehat{D}/W_L - \sqrt{\rho}\, \widetilde{Y}}{\sqrt{1-\rho}} \right) =: \widehat{L} \,, \qquad n \to \infty \,,$$

where now $\widehat{D} = t_{\nu_L}^{-1}(p)$. The variance is given by $\mathrm{var}(\widehat{L}) = \widehat{p}_{12} - p^2$ with $\widehat{p}_{12} = t_{\nu_L;\rho}(\widehat{D}, \widehat{D})$.

We now apply the one-factor approach to market risk so that both market and credit risk are systematically described by one and the same single factor $\widetilde{Y}$. To achieve this, we use (4.1) and

$$\tilde{\eta} := \frac{1}{\sqrt{1-\tilde{\gamma}^2}} \left[\sum_{k=1}^{K} \left(\gamma_k - \frac{\tilde{\gamma}}{\sqrt{\rho}} \beta_k \right) Y_k + \sqrt{1 - \sum_{k=1}^{K} \gamma_k^2}\, \eta \right]$$

[1] Actually, there are less restrictive conditions for the exposures e_i and the individual default variables L_i under which the LHP approximation still holds, see, e.g., in Bluhm et al. [3], Section 2.5.1.

with

$$\tilde{\gamma} := \frac{1}{\sqrt{\rho}} \sum_{k=1}^{K} \beta_k \gamma_k \,. \tag{4.4}$$

Then, we obtain the formal identities

$$Z = -\sigma \left(\tilde{\gamma}\, \widetilde{Y} + \sqrt{1-\tilde{\gamma}^2}\, \tilde{\eta} \right) \tag{4.5}$$

and

$$\widehat{Z} = -\sigma\, W_Z \left(\tilde{\gamma}\, \widetilde{Y} + \sqrt{1-\tilde{\gamma}^2}\, \tilde{\eta} \right) \tag{4.6}$$

for the normal factor model (2.10) and for the shock model (2.11), respectively. In both cases, $\tilde{\eta}$ is standard normally distributed and independent of $\widetilde{Y}$. Moreover, Z in (4.5) is normally distributed with zero mean and variance $\mathrm{var}(Z) = \sigma^2$, whereas $\widehat{Z}$ in (4.6) follows a t-distribution with ν_Z degrees of freedom.

While the one-factor weight $\sqrt{\rho}$ for the credit portfolio depends only on the β_k, the one-factor weight $\tilde{\gamma}$ for market risk given by (4.4) is a function of $\beta_k \gamma_k$. In particular, in order to obtain non-vanishing systematic market risk within the one-factor model, both risk types have to share at least one common factor.

4.2 One-Factor Inter-Risk Correlation

The calculations of Section 3 easily apply to the case of the joint one-factor model of credit and market risk and the results regarding inter-risk correlation simplify considerably. We start with the normal one factor model.

Normal Factor Model Approach. Instead of (3.2) we now obtain

$$\mathrm{corr}(L^{(n)}_{\mathrm{hom}}, Z) = \frac{\sqrt{n}\, r\, e^{-D^2/2}}{\sqrt{2\pi}\sqrt{p_{12}(n-1) + p(1-np)}},$$

where $D = \Phi^{-1}(p)$ is the default point, $p_{12} = \Phi_\rho(D, D)$ is the joint default probability within the homogenous portfolio, and $r = \mathrm{corr}(Z, A_i) = \sqrt{\rho}\,\tilde{\gamma} = \sum_{k=1}^{K} \beta_k \gamma_k$. If the credit portfolio is not only homogenous but also very large, we arrive at the following LHP approximation for the inter-risk correlation between the credit portfolio loss (4.3) and market risk P/L (4.5) in the limit $n \to \infty$:

$$\mathrm{corr}(L, Z) = \frac{r\, e^{-D^2/2}}{\sqrt{2\pi(p_{12} - p^2)}} \,. \tag{4.7}$$

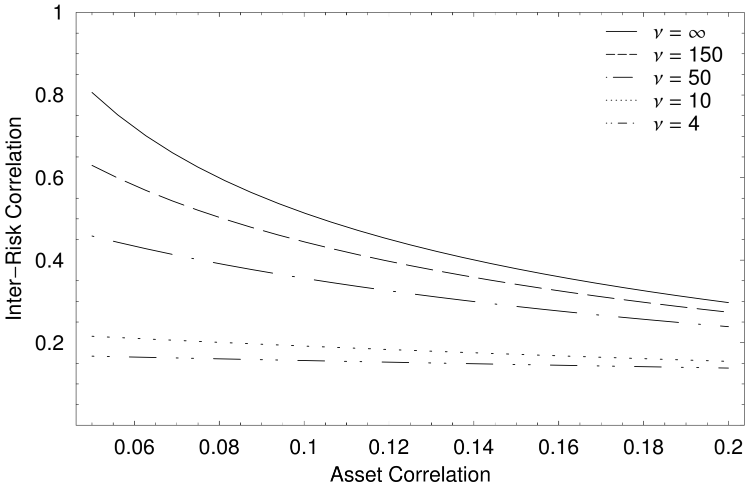


Figure 1: LHP approximations of inter-risk correlation for $r = 0.2$ as a function of the asset correlation ρ according to the normal factor model (equation (4.7)) and the common shock model (equation (4.10)). The average default probability is assumed to be $p = 0.002$.

Given the uniform asset correlation $\rho = \sum_{k=1}^{K} \beta_k^2$ of a homogenous credit portfolio, it follows from (4.4) that $|\tilde{\gamma}| \leq 1$, and thus $|r| \leq \sqrt{\rho}$, implying the bounds

$$|\mathrm{corr}(L, Z)| \leq \frac{\sqrt{\rho}\, e^{-D^2/2}}{\sqrt{2\pi(p_{12} - p^2)}} =: \psi(p, \rho)\,. \tag{4.8}$$

Shock Model Approach. The LHP approximation for inter-risk correlation in the case of the independent shock model of Definition 2.8 (1) yields

$$\mathrm{corr}(\widehat{L}, \widehat{Z}) = \sqrt{\frac{\nu_Z - 2}{2}}\, \frac{\Gamma\left(\frac{\nu_Z - 1}{2}\right)}{\Gamma\left(\frac{\nu_Z}{2}\right)}\, \frac{r\left(1 + \frac{\widehat{D}^2}{\nu_L}\right)^{-\frac{\nu_L}{2}}}{\sqrt{2\,\pi\,(\widehat{p}_{12} - p^2)}}\,, \tag{4.9}$$

whereas for the common shock model of Definition 2.8 (2) we obtain

$$\mathrm{corr}(\widehat{L}, \widehat{Z}) = \sqrt{\frac{\nu - 2}{2}}\, \frac{\Gamma\left(\frac{\nu - 1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\, \frac{r\left(1 + \frac{\widehat{D}^2}{\nu}\right)^{\frac{1-\nu}{2}}}{\sqrt{2\,\pi\,(\widehat{p}_{12} - p^2)}}\,, \tag{4.10}$$

where $\widehat{D} = t_\nu^{-1}(p)$ is the default point, $\widehat{p}_{12} = t_{\nu;\rho}(\widehat{D}, \widehat{D})$ is the joint default probability within the homogenous portfolio, and $r = \mathrm{corr}(\widehat{Z}, \widehat{A}_i) = \sqrt{\rho}\,\tilde{\gamma}$. Similarly as for the LHP approximation of the normal factor model, bounds for the inter-risk correlation can be obtained from (4.9) and (4.10) together with $|r| \leq \sqrt{\rho}$. In the special case that $\nu_L = \nu_Z = \nu$ it follows by comparison of (4.9) and (4.10) that the assumption of one single common

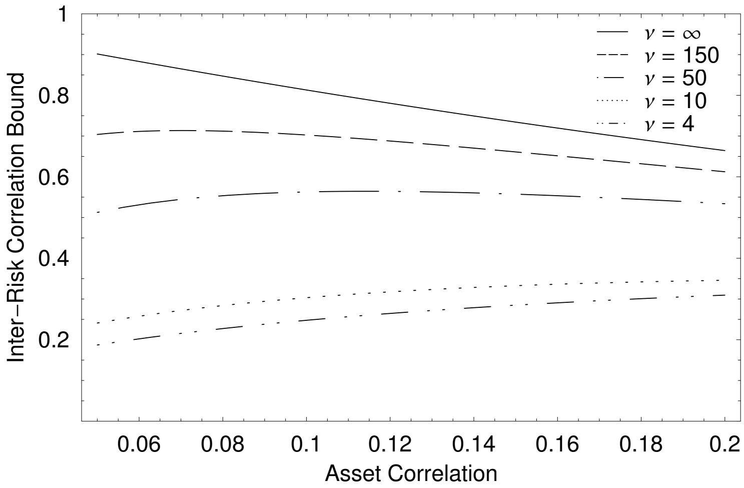


Figure 2: LHP approximations of the inter-risk correlation bound as a function of the asset correlation ρ according to the normal factor model (equation (4.8)) and the common shock model (equation (4.10) with $r = \sqrt{\rho}$). The average default probability is assumed to be $p = 0.002$.

shock increases inter-risk correlation by a factor of $\sqrt{1 + \frac{\widehat{D}^2}{\nu}}$ compared to the independent shock model. For typical values of p, this factor lies in a range of about 1.0–2.0.

To find out how global (macroeconomic) shocks affect inter-risk correlation, let us contrast the LHP approximations (4.9) and (4.10) of the shock models with that of the normal factor model (4.7). For this purpose, Table 4.1 as well as Figures 1 and 2 compare the inter-risk correlation and its upper bound for the common shock model with the outcome of the normal factor model. One can see that the common shock model yields—particularly for small asset correlations—lower inter-risk correlations and bounds than the normal factor model. In the case of the independent shock model, the spread between the normal inter-risk correlation and the shocked inter-risk correlation would be even higher.

Needless to say, the one-factor asset correlation ρ is a popular parameter in the context of credit portfolio modelling. It is often used as a "single-number measure" to evaluate the average dependence structures between different counterparties of a credit portfolio. Moreover, it plays an important role also in the calculation formula for regulatory capital charges according to the *internal-ratings-based* (IRB) approach of Basel II [1]. Equations (4.8)–(4.10) now show that ρ is a very important parameter also beyond credit risk itself as it determines its maximum possible inter-risk correlation with another risk type, here market risk; see again Figure 2 where inter-risk correlation bounds are plotted as a function of ρ^2.

[2]Note that ρ enters $|\mathrm{corr}(L, Z)|$ not only directly by $\sqrt{\rho}$ but also indirectly via the joint default

ρ	Normal Model $\nu = \infty$	Common Shock Model $\nu = 4$	$\nu = 10$	$\nu = 50$
		$p = 0.002$		
5 %	0.81 (0.90)	0.17 (0.19)	0.22 (0.24)	0.46 (0.51)
10 %	0.51 (0.81)	0.16 (0.25)	0.19 (0.30)	0.36 (0.56)
15 %	0.38 (0.73)	0.15 (0.28)	0.17 (0.33)	0.29 (0.56)
20 %	0.30 (0.66)	0.14 (0.31)	0.15 (0.35)	0.24 (0.53)
		$p = 0.02$		
5 %	0.85 (0.95)	0.27 (0.31)	0.37 (0.42)	0.62 (0.70)
10 %	0.57 (0.90)	0.25 (0.40)	0.33 (0.52)	0.48 (0.76)
15 %	0.44 (0.86)	0.24 (0.46)	0.29 (0.57)	0.39 (0.76)
20 %	0.37 (0.82)	0.22 (0.50)	0.27 (0.59)	0.33 (0.75)

Table 4.1: LHP approximation for inter-risk correlation for the normal factor model (4.7) and the common shock model (4.10) using $r = 0.2$ but different values for p and average asset correlation ρ. The values in brackets indicate upper inter-risk correlation bounds for which $r = \sqrt{\rho}$.

Similarly, for a fixed uniform asset correlation ρ, inter-risk correlation and its bound depend on the average default probability p and thus on the average rating of the credit portfolio. This is depicted in Figure 3 where LHP inter-risk correlation as well as the corresponding upper bounds are plotted as a function of the average portfolio rating. One can see that an improvement of the average portfolio rating structure decreases inter-risk correlation (as well as its bounds) and thus tends to result in a lower volatility of the total portfolio of market and credit.

A Moment Estimator for the Inter-Risk Correlation Bound. Even if the actual credit portfolio is not homogenous, the derived LHP approximation provides us with a useful estimator for approximating the upper inter-risk correlation bound.

Let us consider the normal factor model and so expression (4.8). For any credit loss distribution—obtained for instance by Monte Carlo simulation— estimators $\hat{p}$ and $\hat{\rho}$ for p and ρ, respectively, can be (numerically) obtained via moment matching. In doing so, we compare the empirical expected loss μ and the empirical variance ς^2 derived from the simulated credit portfolio with the corresponding figures of an LHP approximation, i.e., average default probability p and variance $\operatorname{var}(L)$. Thus we require that

$$\mu = e_{\text{tot}}\, \hat{p} \tag{4.11}$$

probability $p_{12} = \Phi_\rho(\Phi^{-1}(p), \Phi^{-1}(p))$. This implies that $|\operatorname{corr}(L, Z)| \neq 0$ for $\rho \to 0$.

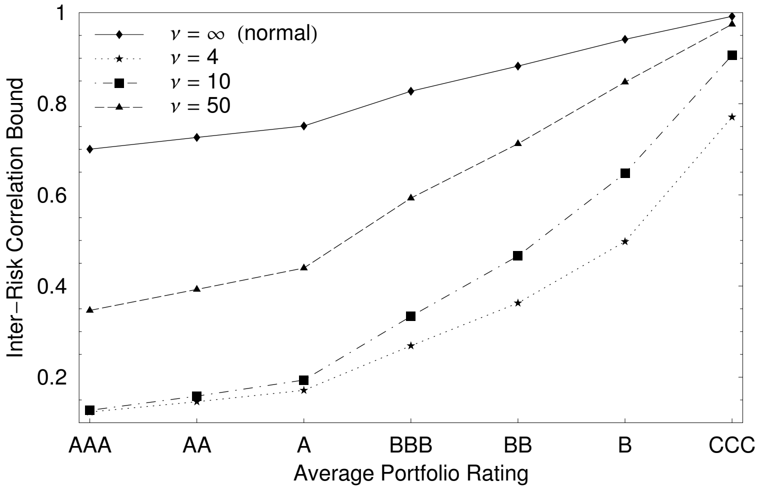


Figure 3: LHP approximations of inter-risk correlation bound as a function of the average portfolio rating according to the normal factor model (equation (4.8)) and the common shock model (equation (4.10) with $r = \sqrt{\rho}$). The average asset correlation is assumed to be $\rho = 10$ %.

and

$$\begin{aligned} \varsigma^2 &= e_{\text{tot}}^2 \,(p_{12} - \hat{p}^2) \\ &= e_{\text{tot}}^2 \left[\Phi_{\hat{\rho}}(\Phi^{-1}(\hat{p}), \Phi^{-1}(\hat{p})) - \hat{p}^2 \right], \end{aligned} \tag{4.12}$$

where e_{tot} denotes the total credit exposure. From (4.8) we then obtain the following moment estimator for the upper inter-risk correlation bound:

$$\hat{\psi}(\hat{p}, \hat{\rho}) = \frac{e_{\text{tot}}}{\varsigma} \, \frac{\sqrt{\hat{\rho}} \, \exp\left[-\frac{1}{2}\big(\Phi^{-1}(\hat{p})\big)^2 \right]}{\sqrt{2\pi}} \,. \tag{4.13}$$

For instance, for the credit test portfolio described in Appendix 7.1 we have $e_{\text{tot}}/\varsigma = 92.41$, $\hat{p} = 0.54$ %, $\hat{\rho} = 23.31$ %, and (4.13) yields $\hat{\psi} = 0.69$. In contrast, the exact bound for the inter-risk correlation given by expression (3.3) evaluates to 0.57.

5 Risk Aggregation

As we already mentioned in the introduction, the estimation of aggregated economic capital is a key element both for regulatory and bank internal purposes. Usually economic capital is defined as a quantile-based risk measure that only reflects unexpected potential loss. More precisely, assume that a certain risk type is represented by a random variable X_i with distribution function $F_i(x) = \mathbb{P}(X_i \leq x)$. If the expectation of X_i exists, we define its economic capital at confidence level α as

$$\mathrm{EC}_i(\alpha) = F_i^{\leftarrow}(\alpha) - \mathbb{E}(X_i)\,,$$

where $F_i^{\leftarrow}(\alpha) = \inf\{x \in \mathbb{R} : F_i(x) \geq \alpha\}$, $0 < \alpha < 1$, is the generalized inverse of F_i. If F_i is strictly increasing and continuous, we may write $F_i^{\leftarrow}(\cdot) = F_i^{-1}(\cdot)$.

The joint factor models proposed here allow us to compare some of the most important approaches for risk aggregation that are used in practice and discussed in the literature.

First, most straight forward is clearly the simple summation of pre-aggregated risk figures obtained for each risk type. Though it typically overestimates total risk, it is still used in practice. Second, aggregated risk capital can be obtained by a joint Monte Carlo simulation of the factor Y as well as the idiosyncratic factors ε_i and η entering both (2.2) and (2.10). Finally, the last two techniques we want to mention here are the copula approach and the square-root-formula approach, which shall be considered in greater detail below. According to The IFRI/CRO Forum [7], the square-root formula approach is most popular in the banking industry while copula methods are often used in insurance. In the sequel we restrict ourselves to the joint normal factor model for credit and market risk given by Definition 2.6.

Square-Root-Formula Approach. Though mathematically justified only in the case of elliptically (e.g., multivariate normally) distributed risk types, this approach is often used in practice because risk-type aggregation can be achieved without simulation by means of a closed-form expression. In our bivariate case of a credit portfolio $L^{(n)}$ and (pre-aggregated) market portfolio Z, the square-root formula (1.1) reads

$$\mathrm{EC}_{L^{(n)}+Z}(\alpha) = \sqrt{\mathrm{EC}_{L^{(n)}}(\alpha)^2 + \mathrm{EC}_Z(\alpha)^2 + 2\,\mathrm{corr}(L^{(n)}, Z)\,\mathrm{EC}_{L^{(n)}}(\alpha)\,\mathrm{EC}_Z(\alpha)}\,,$$

where $\mathrm{corr}(L^{(n)}, Z)$ is the inter-risk correlation (3.2).

Copula Aggregation Approach. A d-dimensional distributional copula C is a d-dimensional distribution function on $[0,1]^d$ with uniform marginals. The relevance of distributional copulas for risk integration is mainly because of Sklar's theorem, which states that for a given copula C and (continuous) marginal distribution functions $F_1, \ldots, F_d$, the joint distribution can be obtained via

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)). \tag{5.1}$$

Therefore, if one has specified marginal distributions for each risk type together with an appropriate copula, total aggregate loss can be obtained numerically or by Monte Carlo simulation.

With regard to the copula technique, an apparent problem is which copula to choose and how to calibrate the model. A remarkable feature of the joint normal factor model for credit and market risk is that it can easily be interpreted as a Gaussian coupling model between market and credit risk.

$\tilde{\gamma}$	0.0	0.2	0.4	0.6	0.8	1.0
$\operatorname{corr}(L, Z)$	0.0	0.15	0.29	0.44	0.59	0.73

Table 5.2: Relation between inter-risk correlation $\operatorname{corr}(L, Z)$ and Gaussian copula parameter $\tilde{\gamma}$ according to (5.2) for $p = 0.002$ and $\rho = 15$ %.

Proposition 5.1 (Normal one-factor model and Gaussian copula)**.** *Consider the joint normal one-factor model for credit and market risk, i.e., we consider* (4.2) *and* (4.5) *where all idiosyncratic factors ε_i of the credit model are independent the idiosyncratic factors $\tilde{\eta}$ of market risk. Then*

(1) *both risk types are coupled by a Gaussian copula with parameter $\tilde{\gamma}$ given by* (4.4).

(2) *the copula parameter $\tilde{\gamma}$ and the inter-risk correlation* $\operatorname{corr}(L, Z)$ *are related by*

$$\tilde{\gamma} = \frac{\operatorname{corr}(L, Z)}{\psi} \tag{5.2}$$

where ψ is the LHP approximation (4.8) *for the inter-risk correlation bound.*

It follows from (5.2) that the absolute value of the inter correlation between credit and market risk is always below the absolute value of the copula parameter $\tilde{\gamma}$. Furthermore, maximum inter-risk correlation corresponds to $\tilde{\gamma} = 1$ for which market risk is completely determined by one single risk factor without having any idiosyncratic component, cf. equation (4.5). A numerical example for (5.2) is given in Table 5.

Estimators for the Gaussian Copula Parameter. Particularly important for practical applications is the question of how the Gaussian copula parameter can be estimated for general credit portfolios. Note that in this case Proposition 5.1 (1) is not directly applicable because β_k in (4.4) is only defined for a homogenous portfolio. However, we can extend the LHP approximation for a credit portfolio, which we have used to construct the estimator $\hat{\psi}$ for the inter-risk correlation bound given by (4.13), to the joint one-factor risk model of credit *and* market risk by matching the inter-risk correlations. If market and credit risk are described by the joint normal factor model of Definition 2.6, we can calculate inter-risk correlation by Theorem 3.1 and compare it to the result in the case of the LHP approximation, i.e., expression 4.7. Then, using Proposition 5.1 (2), we arrive at the following general estimator for the copula parameter $\tilde{\gamma}$,

$$\widehat{\tilde{\gamma}}_1 = \frac{\widehat{\operatorname{corr}}(L, Z)}{\hat{\psi}} \tag{5.3}$$

where

$$\widehat{\operatorname{corr}}(L, Z) = \operatorname{corr}(L^{(n)}, Z)$$

with $\mathrm{corr}(L^{(n)}, Z)$ calculated as in (3.2).

An alternative estimator for $\tilde{\gamma}$ can be constructed by applying the right-hand side of (5.2) directly to a non-homogenous portfolio without introducing a one-factor approximation before. In this case it follows together with (3.2) and (3.3) that

$$\widehat{\tilde{\gamma}}_2 = \frac{\sum_{i=1}^{n} \sum_{k=1}^{K} \beta_{ik}\, \gamma_k\, e_i \exp\left(-\frac{1}{2} D_i^2\right)}{\sum_{i=1}^{n} e_i \sqrt{\sum_{k=1}^{K} \beta_{ik}^2}\, \exp\left(-\frac{1}{2} D_i^2\right)} \,. \tag{5.4}$$

We now illustrate our findings by means of the sample portfolio of credit and market risk described in Appendix 7.1. For the estimators above as well as the inter-risk correlation (3.2) we then obtain $\widehat{\tilde{\gamma}}_1 = 0.32$, $\widehat{\tilde{\gamma}}_2 = 0.39$, and $\mathrm{corr}(L^{(n)}, Z) = 0.22$. These parameters can now be used for risk aggregation as described before. Results for aggregated economic capital at different confidence levels α are summarized in Table 5.3. Some remarks are appropriate. In the first two rows of Table 5.3, stand-alone credit risk and market risk were calculated by the general models of Definitions 2.1 and 2.5, respectively. These figures were directly used in the square-root formula approach. In contrast, for copula aggregation the marginal distribution function for credit risk was first approximated by a one-factor model using moment matching (4.11) and (4.12). Finally, the copula parameter was estimated via the moment estimator $\widehat{\tilde{\gamma}}_1$.

EC	$\alpha = 0.9$	$\alpha = 0.99$	$\alpha = 0.999$	$\alpha = 0.9998$
Credit	0.16	0.87	1.91	2.68
Market	0.23	0.42	0.56	0.64
Sum	0.39	1.29	2.47	3.32
Square-root formula ($\mathrm{corr}(L^{(n)}, Z) = 0.22$)	0.31	1.04	2.10	2.89
Copula approach ($\tilde{\gamma} = 0.32$)	0.32	1.03	2.20	3.16
Simulation approach	0.32	1.04	2.09	2.89

Table 5.3: Aggregated economic capital in EUR bn for different confidence levels α obtained by the four aggregation methods described in the text.

In this particular example, the square-root formula seems to be a reasonable proxy when economic capital at high confidence level is considered. The copula approach leads to a heavier tailed loss distribution estimate resulting in more conservative quantile estimates.

6 Conclusion

The model we have proposed here extends a classical structural portfolio model for credit loss to a joint linear model for both credit and market (or other) risk. This enables us to calculate inter-risk correlation between credit and market risk analytically, and to derive upper bounds for inter-risk correlation, which can be applied in the absence of any specific information regarding the market risk portfolio. Moreover, we have suggested a moment estimator for the inter-risk correlation bound that works for almost arbitrary simulated credit loss distributions. Hence, our findings are of utmost importance for economic capital aggregation, in particular in the context of Pillar II compliance.

Our approach turns out to be quite flexible in the sense that also typical heavy tail characteristics of market as well as credit risk can easily be included by allowing the underlying factors to be t-distributed. This enables the risk manager to investigate how inter-risk correlation between credit and market risk is potentially impacted by model risk.

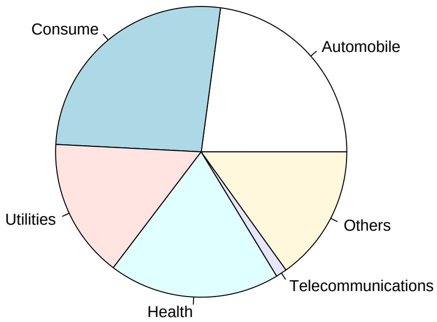
We have used the results obtained for the inter-risk correlation to numerically explore risk aggregation of a credit and market risk portfolio using different popular methodologies. First, adopting an LHP approximation, we have shown how a Gaussian coupling model between the two risk types can explicitly be parameterized and estimated. Hence, our approach is an important step forward towards a reliable and feasible method for economic capital aggregation that can be implemented in practical risk measurement without greater obstacles.

We then have compared the copula based technique with the popular square-root formula (or variance-covariance method) for risk type aggregation. The latter can be easily applied here because we have obtained inter-risk correlation analytically. Needless to say, since the marginal distributions of different risk types in general do not belong to the class of elliptical distributions, the square-root-formula approach has to be applied cautiously. Interestingly, in the normal factor model, it matches quite accurately with the aggregated risk of our sample portfolio when high confidence levels are considered. We made similar observations in the case of other sample portfolios, and it might be interesting to investigate whether this is the consequence of a more general result.

Our approach could be extended in several ways. First, it could be examined whether the idea of closed-form expressions for the inter-risk correlation could be applied also to other risk types (instead of market risk) that are neither normal nor t-distributed, such as operational risk as a prominent example. Another interesting question is how our results would change if one would switch from the default-mode credit model we have used here to a fair-value approach, i.e, if one would allow for rating migrations.

From our discussions we have had so far with colleagues from the banking industry, we feel that there is still room for investigation with regard to inter-risk correlation. Here, we have focused on credit and market risk, and we would like encourage the economic capital community to share their experience, feedback and findings.

Figure 4: Histogram of the exposure sizes in the credit portfolio logarithmic scale.



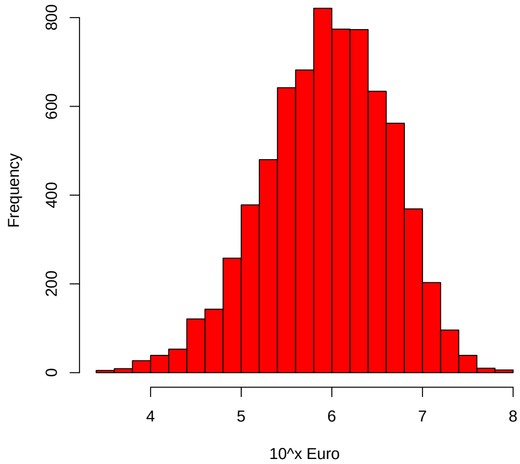


Figure 5: Relative exposures of the industry sectors in the sample credit portfolio.

7 Appendix

7.1 The Test Portfolios of Market and Credit Risk

The sample credit portfolio consists of 7,124 loans with a total exposure of ca. 18 billion Euro. The obligors are assigned to seven different industry sectors such as car industry or telecommunication. The single exposures range between 3,200 and 82 million Euro. The default probabilities are given with an average default probability of 2.08 percent and an exposure-weighted average default probability of 1.21 percent. Loss given default is set to 45 percent.

The standard deviation σ of the market portfolio, represented by the single random variable Z, is 180 million Euro.

In the normal factor approaches, dependence between the asset log returns $A_i, i = 1, \dots, n$, of the obligors and the market loss variable Z is due to the systematic factors $Y = (Y_1, \dots, Y_K)$. The corresponding factor loadings β_{ik} and γ_k are estimated by Maximum-Likelihood factor analysis, see, e.g., Fahrmeir et al. [6]. However, since we cannot observe the A_i directly, we assume that the dependence between the corresponding industry sector stock indices is similar and hence we use them for the estimation of the factor loadings.

For the market loss variable, we use P/L figures as they typically occur in trading business. The data range is between February 2002 and March 2006.

7.2 Proofs

Proof of Theorem 3.1. First we calculate the covariance between $L^{(n)}$ and Z. Using $\mathbb{E}(Z) = 0$, expression (2.1), and the fact that η in (2.10) is independent from Y (and thus from L_i), we can write

$$\operatorname{cov}(L^{(n)}, Z) = \mathbb{E}(ZL^{(n)}) = -\sigma \sum_{i=1}^{n} e_i \sum_{k=1}^{K} \gamma_k \, \mathbb{E}(Y_k L_i) \,. \tag{7.5}$$

To evaluate the expectation, conditioning with respect to $Y_k = y_k$ and using the law of iterated expectation yield

$$\begin{aligned}
\mathbb{E}(Y_k \, L_i) &= \mathbb{E}(Y_k \, L_i(Y_1, \ldots, Y_K)) \\
&= \mathbb{E}\big(\mathbb{E}(Y_k \, L_i(Y_1, \ldots, Y_K) \,|\, Y_k)\big) \\
&= \int_{-\infty}^{\infty} \mathbb{E}(Y_k \, L_i(Y_1, \ldots, Y_K) \,|\, Y_k = y_k) \; d\Phi(y_k) \\
&= \int_{-\infty}^{\infty} \mathbb{E}(y_k \, L_i(Y_1, \ldots, y_k, \ldots, Y_K)) \; d\Phi(y_k) \\
&= \int_{-\infty}^{\infty} y_k \, \mathbb{E}(L_i(Y_1, \ldots, y_k, \ldots, Y_K)) \; d\Phi(y_k)
\end{aligned}$$

where Φ is the standard normal distribution function. Using $\mathbb{E}(L_i) = \mathbb{P}(L_i = 1)$, we have

$$\begin{aligned}
\mathbb{E}(Y_k \, L_i) &= \int_{-\infty}^{\infty} y_k \, \mathbb{P}(L_i(Y_1, \ldots, y_k, \ldots, Y_K) = 1) \; d\Phi(y_k) \\
&= \int_{-\infty}^{\infty} y_k \, \mathbb{P}\left(\sum_{\substack{l=1 \\ l \neq k}}^{K} \beta_{il} \, Y_l + \beta_{ik} \, y_k + \sqrt{1 - \sum_{j=1}^{K} \beta_{ij}^2} \; \varepsilon_i \leq D_i \right) d\Phi(y_k) \\
&= \int_{-\infty}^{\infty} y_k \, \mathbb{P}\left(\sum_{\substack{l=1 \\ l \neq k}}^{K} \beta_{il} \, Y_l + \sqrt{1 - \sum_{j=1}^{K} \beta_{ij}^2} \; \varepsilon_i \leq D_i - \beta_{ik} \, y_k \right) d\Phi(y_k) \\
&=: \int_{-\infty}^{\infty} y_k \, \mathbb{P}\left(X \leq D_i - \beta_{ik} \, y_k\right) d\Phi(y_k)
\end{aligned}$$

where X is normally distributed with variance $\operatorname{var}(X) = 1 - \beta_{ik}^2$. Hence, we obtain

$$\mathbb{E}(Y_k \, L_i) \;=\; \int_{-\infty}^{\infty} y_k \, \Phi\left(\frac{D_i - \beta_{ik} \, y_k}{\sqrt{1 - \beta_{ik}^2}} \right) d\Phi(y_k) \,.$$

Since the derivative of the density φ of the standard normal distribution is given by $\varphi'(y) = y\,\varphi(y)$, it follows by partial integration that

$$\mathbb{E}(Y_k\, L_i) \;=\; -\frac{\beta_{ik}}{\sqrt{1-\beta_{ik}^2}} \int_{-\infty}^{\infty} y_k\, \varphi\left(\frac{D_i - \beta_{ik}\, y_k}{\sqrt{1-\beta_{ik}^2}}\right) \varphi(y_k)\, d\varphi(y_k)\,,$$

where the right-hand side is just $-\beta_{ik}$ times the density of a random variable $\Psi = \sqrt{1-\beta_{ik}^2}X + \beta_{ik}Y_k$ for standard normal iid X, Y_k at point D_i. Since Ψ is then again standard normal, we obtain

$$\mathbb{E}(Y_k\, L_i) \;=\; -\beta_{ik}\varphi(D_i) \;=\; -\frac{\beta_{ik}}{\sqrt{2\pi}}\, e^{-\frac{D_i^2}{2}}\,, \tag{7.6}$$

which together with (7.5) yields

$$\begin{aligned}\operatorname{cov}(L^{(n)}, Z) \;&=\; \frac{\sigma}{\sqrt{2\pi}} \sum_{i=1}^{n} \sum_{k=1}^{K} e_i\, \gamma_k\, \beta_{ik}\, e^{-\frac{D_i^2}{2}} \\ &=\; \frac{\sigma}{\sqrt{2\pi}} \sum_{i=1}^{n} e_i\, r_i\, e^{-\frac{D_i^2}{2}}\end{aligned}$$

where we have introduced $r_i := \operatorname{corr}(A_i, Z) = \sum_{k=1}^{K} \beta_{ik}\gamma_k$. With $\sqrt{\operatorname{var}(Z)} = \sigma$ and

$$\begin{aligned}\operatorname{var}(L^{(n)}) \;&=\; \sum_{i,j=1}^{n} e_i\, e_j \operatorname{cov}(L_i, L_j) \\ &=\; \sum_{i,j=1}^{n} e_i\, e_j \left(\mathbb{E}(L_i L_j) - \mathbb{E}(L_i)\mathbb{E}(L_j)\right) \\ &=\; \sum_{i,j=1}^{n} e_i\, e_j\, (p_{ij} - p_i\, p_j)\,,\end{aligned} \tag{7.7}$$

where p_{ij} is the joint default probability (2.5), the assertion follows. □

Proof of Proposition 3.2. Since the obligor's exposures are assumed to be positive, $e_i \geq 0$, it follows from (3.2) that

$$|\operatorname{corr}(L^{(n)}, Z)| \leq \frac{\sum_i e_i\, |r_i| \exp\left(-\frac{1}{2}D_i^2\right)}{\sqrt{2\pi \sum_{ij} e_i\, e_j\, (p_{ij} - p_i\, p_j)}}\,.$$

From the Cauchy-Schwarz inequality together with $\sum_{k=1}^{K} \gamma_k^2 \leq 1$, it follows that

$$|r_i| = \left|\sum_{k=1}^{K} \beta_{ik}\gamma_k\right| \leq \left(\sum_{k=1}^{K} \beta_{ik}^2\right)^{1/2} \left(\sum_{k=1}^{K} \gamma_k^2\right)^{1/2} \leq \left(\sum_{k=1}^{K} \beta_{ik}^2\right)^{1/2} \leq 1\,,$$

which completes the proof. □

Proof of Theorem 3.3.

(1) Using (2.1) and the law of iterated expectation, we obtain

$$\operatorname{cov}(\widehat{L}^{(n)}, \widehat{Z}) = \sum_{i=1}^{n} e_i \, \mathbb{E}(\widehat{Z}\, \widehat{L}_i) = \sum_{i=1}^{n} e_i \, \mathbb{E}\big(\mathbb{E}(\widehat{Z}\, \widehat{L}_i \,|\, W_L, W_Z)\big). \tag{7.8}$$

Now, in the credit shock model of Definition 2.3 we have that for each loss variable $\widehat{L}_i$

$$\begin{aligned}
\mathbb{P}(\widehat{L}_i = 1) &= \mathbb{P}\left(W_L \sum_{k=1}^{K} \beta_{ik} \, Y_k + W_L \sqrt{1 - \sum_{j=1}^{K} \beta_{ij}^2} \; \varepsilon_i \le \widehat{D}_i \right) \\
&= \mathbb{P}\left(\sum_{k=1}^{K} \beta_{ik} \, Y_k + \sqrt{1 - \sum_{j=1}^{K} \beta_{ij}^2} \; \varepsilon_i \le \frac{\widehat{D}_i}{W_L} \right).
\end{aligned}$$

Hence, the shock factor model conditional on the shock variable W_L is equivalent to a normal factor model with adjusted default points $D_i^* := \widehat{D}_i / W_L$. Therefore, we obtain from (7.6) without any further calculation

$$\begin{aligned}
\mathbb{E}(\widehat{Z}\, \widehat{L}_i \,|\, W_L, W_Z) &= -\sigma \sum_{k=1}^{K} \gamma_k \, W_Z \, \mathbb{E}(Y_k \, \widehat{L}_i \,|\, W_L) \\
&= -\sigma \sum_{k=1}^{K} \gamma_k \, W_Z \left(-\frac{\beta_{ik}}{\sqrt{2\pi}} \, e^{-\frac{D_i^{*2}}{2}} \right) \\
&= \frac{\sigma \, r_i}{\sqrt{2\pi}} \, W_Z \, e^{-\frac{D_i^{*2}}{2}},
\end{aligned}$$

where $r_i = \sum_{k=1}^{K} \beta_{ik} \gamma_k$ for $i = 1, \dots, n$. Integration over W_L and W_Z yields

$$\begin{aligned}
\mathbb{E}\big(\mathbb{E}(\widehat{Z}\, \widehat{L}_i \,|\, W_L, W_Z)\big) &= \frac{\sigma \, r_i}{\sqrt{2\pi}} \int_0^\infty W_Z \, dF_{\nu_Z}(s) \int_0^\infty e^{-\frac{D_i^{*2}}{2}} \, dF_{\nu_L}(s) \\
&= \frac{\sigma \, r_i}{\sqrt{2\pi}} \, \mathbb{E}(W_Z) \int_0^\infty e^{-\frac{\widehat{D}_i^2}{2\nu_L} s} \, f_{\nu_L}(s) \, ds
\end{aligned} \tag{7.9}$$

where F_ν is the distribution function of a χ^2_ν distributed random variable with density $f_\nu(s)$. By substitution, we can perform the integration for $\nu_L > 0$,

$$\begin{aligned}
\int_0^\infty e^{-\frac{\widehat{D}_i^2}{2\nu_L} s} \, f_{\nu_L}(s) \, ds &= \int_0^\infty \frac{2^{-\nu/2} \, s^{\nu/2 - 1}}{\Gamma\left(\frac{\nu}{2}\right)} \, \exp\left[-\left(1 + \frac{\widehat{D}_i^2}{\nu_L} \right) \frac{s}{2} \right] ds \\
&= \left(1 + \frac{\widehat{D}_i^2}{\nu_L} \right)^{-\frac{\nu_L}{2}} \int_0^\infty \frac{2^{-\nu/2} \, e^{-s/2} \, s^{\nu/2 - 1}}{\Gamma\left(\frac{\nu}{2}\right)} \, ds \\
&= \left(1 + \frac{\widehat{D}_i^2}{\nu_L} \right)^{-\frac{\nu_L}{2}}.
\end{aligned}$$

Together with

$$\mathbb{E}(W_Z) = \sqrt{\frac{\nu_Z}{2}}\,\frac{\Gamma\left(\frac{\nu_Z-1}{2}\right)}{\Gamma\left(\frac{\nu_Z}{2}\right)}$$

for $\nu_Z > 1$ we obtain

$$\mathbb{E}\big(\mathbb{E}(\widehat{Z}\,\widehat{L}_i\,|\,W_L, W_Z)\big) \;=\; \frac{\sigma\, r_i}{\sqrt{2\pi}}\sqrt{\frac{\nu_Z}{2}}\left(1+\frac{\widehat{D}_i^2}{\nu_L}\right)^{\frac{-\nu_L}{2}}\frac{\Gamma\left(\frac{\nu_Z-1}{2}\right)}{\Gamma\left(\frac{\nu_Z}{2}\right)}, \tag{7.10}$$

Now, plugging (7.10) into (7.8), and using (7.7) together with

$$\operatorname{var}(\widehat{Z}) = \left(\frac{\nu_Z}{\nu_Z-2}\right)\sigma^2\,, \quad \nu_Z > 2\,,$$

finally leads to

$$\operatorname{corr}(\widehat{L}^{(n)}, \widehat{Z}) = \sqrt{\frac{\nu_Z-2}{2}}\,\frac{\Gamma\left(\frac{\nu_Z-1}{2}\right)}{\Gamma\left(\frac{\nu_Z}{2}\right)}\,\frac{\sum_{i=1}^n e_i\, r_i\left(1+\frac{\widehat{D}_i^2}{\nu_L}\right)^{\frac{-\nu_L}{2}}}{\sqrt{2\,\pi\,\operatorname{var}(\widehat{L}^{(n)})}}\,.$$

(2) Similarly to the case of independent shock variables, we are now conditioning on the single shock variable W. Instead of (7.9), we obtain for $\nu > 1$ by substitution

$$\begin{aligned}
\mathbb{E}\big(\mathbb{E}(\widehat{Z}\,\widehat{L}_i\,|\,W)\big) &= \frac{\sigma\, r_i}{\sqrt{2\pi}}\int_0^\infty W\, e^{-\frac{D_i^{*2}}{2}}\, dF_\nu(s) \\
&= \frac{\sigma\, r_i}{\sqrt{2\pi}}\int_0^\infty \sqrt{\frac{\nu}{s}}\, e^{-\frac{\widehat{D}_i^2}{2\nu}s}\, f_\nu(s)\, ds \\
&= \frac{\sigma\, r_i}{\sqrt{2\pi}}\int_0^\infty \frac{2^{-\nu/2}\,\nu^{1/2}}{\Gamma(\nu/2)} s^{\frac{\nu-1}{2}-1}\exp\left[-\left(\frac{\widehat{D}_i^2}{\nu}+1\right)\frac{s}{2}\right] ds \\
&= \frac{\sigma\, r_i}{\sqrt{2\pi}}\int_0^\infty \frac{2^{-\nu/2}\,\nu^{1/2}}{\Gamma(\nu/2)}\left(\frac{\widehat{D}_i^2}{\nu}+1\right)^{-\frac{\nu-1}{2}} s^{\frac{\nu-1}{2}-1}e^{-s/2}ds \\
&= \frac{\sigma\, r_i}{\sqrt{2\pi}}\,\frac{\nu^{1/2}\,2^{-1/2}}{\Gamma(\nu/2)}\left(\frac{\widehat{D}_i^2}{\nu}+1\right)^{\frac{1-\nu}{2}}\Gamma\left(\frac{\nu-1}{2}\right)\int_0^\infty \frac{2^{-\frac{\nu-1}{2}}}{\Gamma\left(\frac{\nu-1}{2}\right)} s^{\frac{\nu-1}{2}-1}e^{-s/2}ds \\
&= \frac{\sigma\, r_i}{\sqrt{2\pi}}\,\frac{\nu^{\frac{\nu}{2}}\left(\widehat{D}_i^2+\nu\right)^{\frac{1-\nu}{2}}\Gamma\left(\frac{\nu-1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{\nu}{2}\right)}\,,
\end{aligned} \tag{7.11}$$

which finally completes the proof. □

Proof of Proposition 3.4. The proof is analogous to the proof of Proposition 3.2.

Proof of Proposition 5.1.
(1) An important characteristic of copulas is their invariance under monotonously increasing transformations. Since the portfolio loss L in the LHP approximation as given by (4.3) is a monotonously increasing function of $-\widetilde{Y}$, it follows that L and Z have the same copula as $-\widetilde{Y}$ and Z. For the latter we know from the one-factor representation of market risk (4.5) that they are bivariate normally distributed with correlation

$$\operatorname{corr}(-\widetilde{Y}, Z) = \tilde{\gamma} \,.$$

Hence, also L and Z are linked by a Gaussian copula with correlation parameter $\tilde{\gamma}$.
(2) This follows directly from (4.7) and (4.8) together with $r = \sqrt{\rho}\,\tilde{\gamma}$.

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