Building ARIMA and ARIMAX Models for Predicting Long-Term Disability Benefit Application Rates in the Public/Private Sectors

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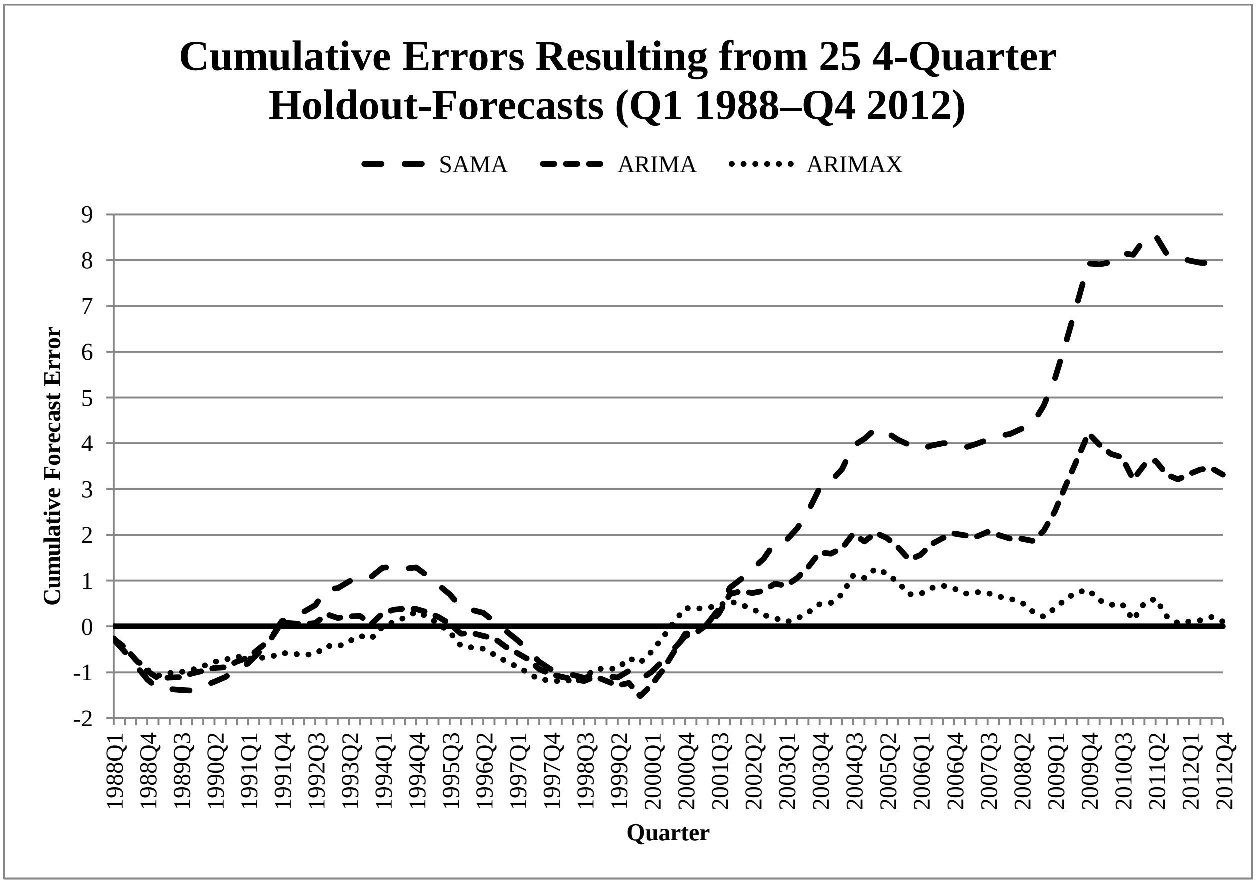
EXECUTIVE SUMMARY

Using the Social Security Disability Insurance benefit claim rate as a proxy, this study investigates two statistical approaches to forecasting long-term disability benefit claims. The results are extendable and should prove useful for insurance carriers who wish to predict short-term future levels of long-term disability benefit claims. The study demonstrates that both the autoregressive integrated moving average (ARIMA) and autoregressive integrated moving average with exogenous variables (ARIMAX) methodologies have the ability to produce accurate four-quarter forecasts.

First built was an ARIMA model, which produces forecasts based upon prior values in the time series (AR terms) and the errors made by previous predictions (MA terms). This typically allows the model to rapidly adjust for sudden changes in trend, resulting in more accurate forecasts. Next built was an ARIMAX model, which is very similar to an ARIMA model, except that it also includes relevant independent variables. While the inclusion of exogenous variables adds complexity to the model-building process, the model can capture the influence of external factors (e.g., the state of the economy) as well as management controllables (e.g., elimination period duration).

The superior performance of both the ARIMA and ARIMAX models against the commonly used seasonally adjusted four-quarter moving average (SAMA) model can be seen in the following graph. Both models' cumulative errors tend to remain close to zero, while the SAMA model's cumulative errors deviate from zero more dramatically. The additional beneficial impact of

including exogenous variables in the model can also be seen by the ARIMAX model's cumulative errors remaining closer to zero.



The benefits to an insurance carrier who is able to accurately predict the disability benefit claims rate are clear. The carrier will be in a much better position to make a wide range of critical planning decisions that are affected by the claims rate, including establishing appropriate reserve levels to service approved claims. This study utilized two powerful techniques to forecast SSDI application rates for benefit claims. Social Security data were chosen primarily because they were readily publically available and familiar to many insurance analysts. However, the model-building exercise detailed in the report can be readily applied to private-sector long-term disability benefit claim application rates.

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ACKNOWLEDGEMENTS

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1. INTRODUCTION

1.1 Purpose of the Study

The Maine Center for Business and Economic Research (MCBER) at the University of Southern Maine, in partnership with the Society of Actuaries (SOA), conducted a predictive model-building exercise to statistically examine and incorporate factors that influence long-term disability (LTD) application rates. This report documents that study. Social Security Administration (SSA) claims-experience data were selected for the model building because they are publicly available and representative (in varying degrees) of the private-sector LTD claims experience. Private LTD carrier data were deemed inappropriate for use in this study because they vary in form, level of detail and their period of collection. Further, it was thought that LTD carriers would find it awkward to share or pool their data with other carriers because they frequently compete in the same markets.

Many of the phenomena that drive Social Security disability application rates are likely to influence LTD application rates for private carriers, which means that exogenous variables that are significant predictors of Social Security Disability Insurance (SSDI) application rates are likely to be strong predictors for private carriers as well. Also, the future experience of at least some private-sector carriers was expected to display a statistical relationship with the application rates projected for Social Security disability.

This study focuses most heavily on the autoregressive integrated moving average with exogenous variables (ARIMAX) methodology, which has the capacity to identify the underlying patterns in time-series data and to quantify the impact of environmental influences. This provides the ARIMAX modeler with the capacity to isolate the influences of high-impact changes of both an external nature (e.g., competitors' activities, the economy and governmental regulations) and an internal nature (e.g., policy coverage, product pricing and target markets). It is also important to note that ARIMAX model building can be reduced/simplified to pure autoregressive integrated moving average (ARIMA) model building if the forecaster/modeler wishes to examine historical behavior and make projections employing only statistically identified historical patterns/relationships.

The target audience for this report is the actuary who either has a basic working knowledge of applied multiple-linear-regression model building or is willing to invest the energy to acquire/recover it. This prerequisite level of understanding of multiple-regression analysis is that which is typically derived from the one or two 3-credit (noncalculus-based) undergraduate courses in applied business statistics required at nationally accredited business schools. As further encouragement for the tentative reader to press forward, the two student co-authors of this report, Bob Swain and Caroline Cole, have completed only the six credits of undergraduate-level statistics required by the business school at the University of Southern Maine.

1.2 Background

To coarsely evaluate the strength of the potential relationship between the application rates for SSDI and those of group LTD carriers, annual data from 2004–10 from 12 of the largest private-sector carriers were examined. Six of the 12 carriers had annual application rates that were significantly correlated ($p \leq 0.10$) with SSDI's annual application rates at lags of 0, 1 and/or 2. Four of these six exhibited one or more significant positive correlations; the other two displayed significant negative correlations, one at lag 1 and the other at lags 1 and 2. (It is important to note that the coarseness of the data and the small sample [n=7] placed serious constraints on this statistical analysis.)

Accurate prediction of future application rates for long-term disability benefits is a major concern for private insurance carriers as well as the Social Security Administration. In both the private and public sectors, the number of claims filed is a key input to many planning decisions. For example, in both sectors, the proper level of reserves required to service approved claims needs to be established, mechanisms to generate revenue streams must be created to maintain appropriate reserve levels, and claims processing and management capacity requirements must be estimated. Unfortunately, application rates are extremely volatile because they are largely driven by forces external to the insurer, be it the SSA in the public sector or an LTD provider in the private sector. For example, at the national level, SSDI applications increased substantially[1] during six of the seven U.S. recessions between 1965 and 2012.[2] Further, a December 2011 article in the *Wall Street Journal,*[3] titled "Jobless Tap Disability Fund," reported on the findings of researchers who have studied the interaction between the condition of the U.S. economy and the SSDI application rate. Some of their findings are summarized below.

- Professor Mark Duggan at the University of Pennsylvania studied the relationship between the U.S. unemployment rate and the application rate for SSDI benefits, and "estimates that the higher unemployment rate [in 2011 compared to 2007] accounts for 3,000 additional people applying for benefits each week."

[1] Social Security Administration, "Disabled Workers."
[2] Wikipedia contributors, "List of recessions in the United States."
[3] Paletta and Searcey, "Jobless Tap Disability Fund."

- Steven Goss, chief actuary of the Social Security Administration, “told Congress … that the 2008–09 recession led to a higher rate of ‘disability incidence’ than any other period except for the economic downturn in 1975.”
- Professor Matthew Rutledge at Boston College studied the relationship between time left until unemployment benefits expire and the likelihood an individual would apply for SSDI benefits, and found that the unemployed are “significantly more likely to apply when [unemployment payments are] ultimately exhausted,” indicating that long-term unemployment is positively linked to the SSDI application rate.
- Massachusetts Institute of Technology professor of economics David Autor summed up his sentiments this way: “To a very large extent, [SSDI] is our big welfare program.”

Some of the other 16 major determinants of the disability application rate listed in *Actuarial Study No. 118* produced by the SSA’s Office of the Actuary[4] include the strength of regional economies, demographic shifts, levels of employment/unemployment and levels of inflation.

1.3 The ARIMAX Methodology

Proper ARIMAX model building has six statistical assumptions that must be addressed and re-addressed as iterative model building progresses. These six assumptions also provide the underpinnings for rigorously performed multiple-regression analysis. While the rules of properly performed regression analysis are rarely fully honored by nonacademic practitioners, when satisfied, they normally lead to much-improved model-building results.

[4] Zayatz, “Social Security Disability.”

Simply put, an ARIMAX model can be viewed as a multiple regression model with one or more autoregressive (AR) terms and/or one or more moving average (MA) terms. Autoregressive terms for a dependent variable are merely lagged values of that dependent variable that have a statistically significant relationship with its most recent value. Moving average terms are nothing more than residuals (i.e., lagged errors) resulting from previously made estimates.

So, for example, a nameless time-series dependent variable y_t might be well estimated by a properly weighted combination of the following four right-hand-side (RHS) variables.

1. x_t = the value of the independent variable x at time t
2. y_{t-1} = the immediately preceding value of the dependent variable y_t at time $t-1$
3. y_{t-2} = the immediately preceding value of the dependent variable y_t at time $t-2$
4. ϵ_{t-4} = the estimation error produced by the model at time $t-4$

This single-independent-variable, multiple-regression-like model for estimating the dependent variable y_t relies on the predictive value of the independent variable x (unlagged), the dependent variable itself (lagged by 1), the dependent variable again (lagged by 2) and a previously produced error term (lagged by 4). That is,

$$\hat{y}_t = \hat{\beta}_1 x_t + \hat{\phi}_1 y_{t-1} + \hat{\phi}_2 y_{t-2} + \hat{\theta}_1 \epsilon_{t-4},$$

where $\hat{\beta}_1$, $\hat{\phi}_1$, $\hat{\phi}_2$ and $\hat{\theta}_1$ are estimated coefficients.

As implied by its shortened acronym, the pure ARIMA model-building methodology employs only lagged values of the dependent variable (i.e., AR terms) and lagged values of errors previously produced by the model (i.e., MA terms). The I in ARIMA refers to integrated and indicates that the dependent variable time series has been differenced one or more times to make the time series stationary before model building begins. (Note: Practically speaking, stationarity implies that the mean and the variance of the time series are not changing over time.) So, for example, the quarterly application rate for SSDI benefits time series used illustratively in this report has displayed a strong overall pattern characterized by both an upward trend and quite-regular quarterly seasonality. As discussed in Section 2.1, to remove the quarterly seasonality, the raw data were differenced by four and then differenced by one to remove the upward trend.

The core difference between formal ARIMAX modeling and the more commonly used multiple regression modeling is that the ARIMAX modeling rigorously adheres to all six of the statistical assumptions underlying regression modeling. Section 2.2 explains these six assumptions. The ARIMAX model-building algorithm flowchart (Figure 8) makes clear the complexity of the iterative process. This level of complexity sometimes discourages model builders from fully adhering to the full set of six key assumptions required for proper regression modeling.

Assumption 3 provides one example of the complexities of meticulously executed regressive modeling in that proper interpretation of the significance levels (p-values) of regression-model coefficients requires that the residuals produced by the model under scrutiny are normally distributed with a mean of zero, a constant variance and (most importantly) with no serial correlation. To satisfy these formal assumptions, it is frequently necessary to model the residuals with ARIMA tools, which often forces originally identified, logically attractive independent variables to lose their significance and to leave the model. This removal of independent variables that appeared to be strong candidates changes the form and character of the residuals and may result in a complete restart of the model-building process.

To address the complex, iterative nature of the ARIMAX model-building process when the pool of explanatory-variable candidates is large, MCBER built a system of integrated SAS[5] software routines to automate the search for the optimal or near-optimal combination of exogenous variables, and AR and MA terms. The resulting ARIMAX models are statistically correct in all regards. Additionally, the composition of both models built using the SAS routines on the illustrative quarterly SSDI-application-rate data set (Q1 1982–Q4 2012) are very intuitively appealing. After differencing by four (to remove seasonality) and then one (to remove trend), the (AR1, AR3, AR10, MA4) ARIMA model produced the best fit with a mean error of 0.005901 and a standard error of 0.0138 for the residuals. The p-values for the coefficients for the AR1, AR3, AR10 and MA4 terms were 0.0002, 0.0035, 0.0045 and < 0.0001, respectively. For the doubly differenced time series, this means that the ARIMA model was built by weighting the most recent actual, the actual three quarters earlier, the actual 10 quarters earlier and the estimation error made four quarters earlier. That is,

$$\hat{y}_t = -0.31102\, y_{t-1} + 0.24543\, y_{t-3} - 0.27565\, y_{t-10} + 0.78139\, \epsilon_{t-4}.$$

The best-fitting ARIMAX model (not coincidentally) has a structure similar to the previous ARIMAX model for the “nameless” dependent variable introduced on Page 7. The AR1 and MA4 terms from the ARIMA model were accompanied by wage-and-salary employment (*wse*) (lag 0) and an AR2 term. That is,

$$\hat{y}_t = -0.000084\text{wse}_t - 0.52665 y_{t-1} - 0.24847 y_{t-2} + 0.79785 \epsilon_{t-4}.$$

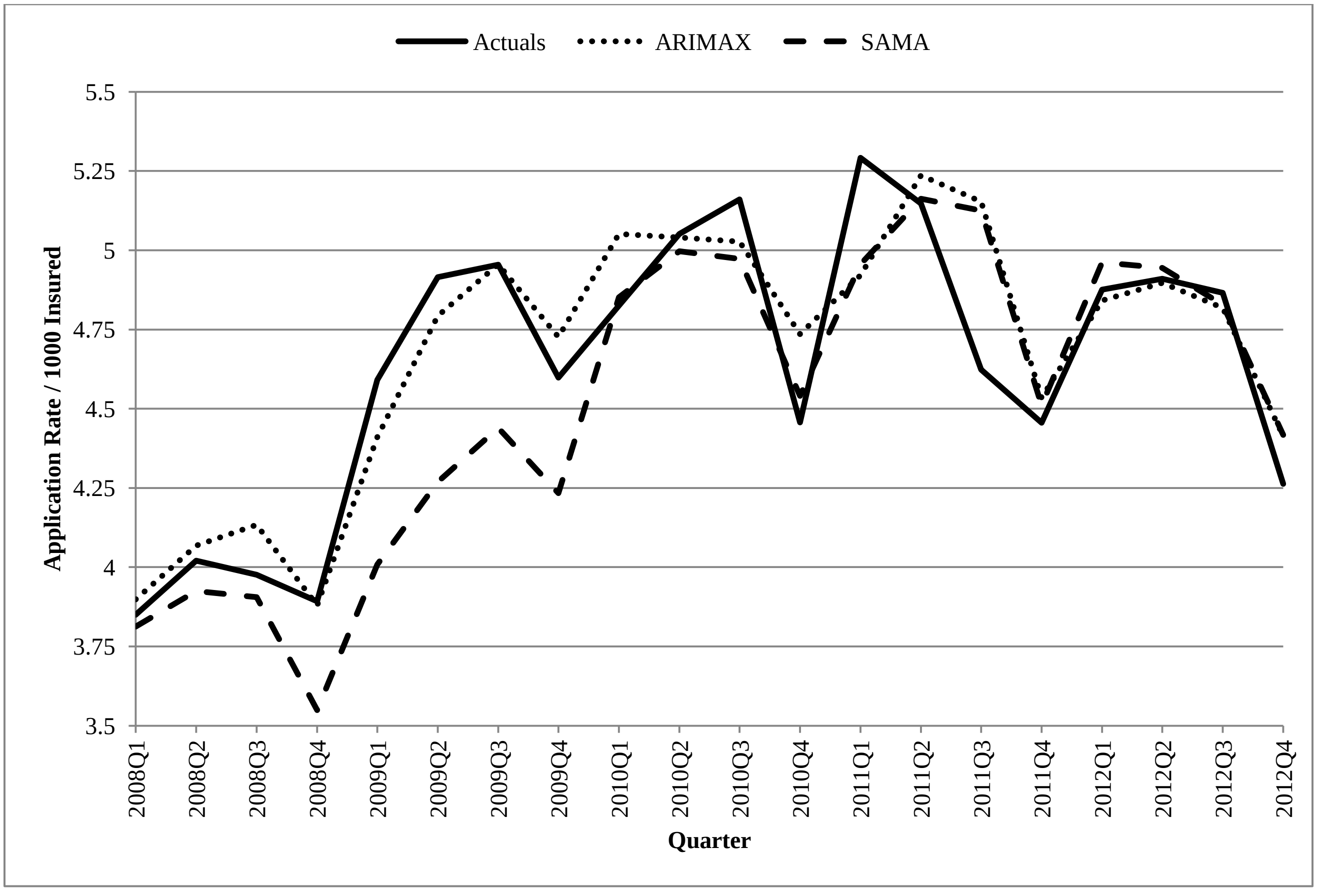
[5] http://www.sas.com

This model produced a mean error of 0.004823 and a standard error of 0.0130. The p-value for the coefficients of the AR1, MA4 and the independent variable *wse* were all ≤ 0.0001, and the p-values for the AR2 term was 0.0058. The fitting and forecasting capacities of the ARIMA and ARIMAX models are discussed in further detail on pages 50-52.

1.4 Comparison of ARIMAX and SAMA Models

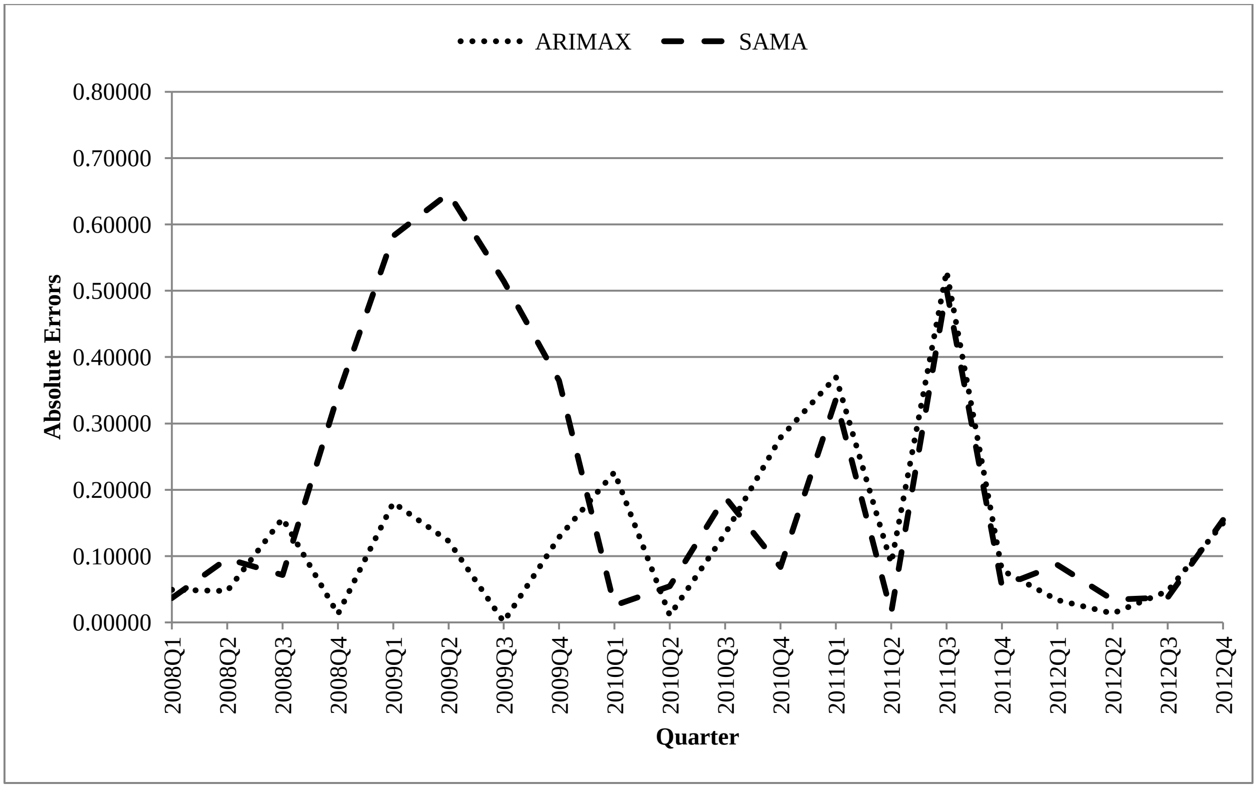
To examine the relative precision of the best-fitting ARIMAX model, its fit performance was compared against that of the commonly used seasonally adjusted four-quarter moving average (SAMA) model. Figure 1 shows the 20 most recent actual quarterly SSDI application rates and the fit estimates produced by each model. The ARIMAX model clearly does a better job of estimating the actual application rates, particularly during periods of steady rising or declining.

Figure 1. A comparison of the ARIMAX and SAMA models' fit estimates with the actual data.



The absolute values of the estimation errors of the two models are compared in Figure 2, which further demonstrates the ARIMAX model's superior precision.

Figure 2. A comparison of the ARIMAX and SAMA models' absolute errors.



The mean absolute percent errors (MAPEs) and mean absolute deviations (MADs) for both models over all 116 quarters for which both models produce estimates (Q1 1984–Q4 2012) and for the most recent 20 quarters (Q1 2008–Q4 2012) are shown in Chart 1.

Chart 1. A comparison of the ARIMAX and SAMA models' goodness-of-fit over two different time horizons.

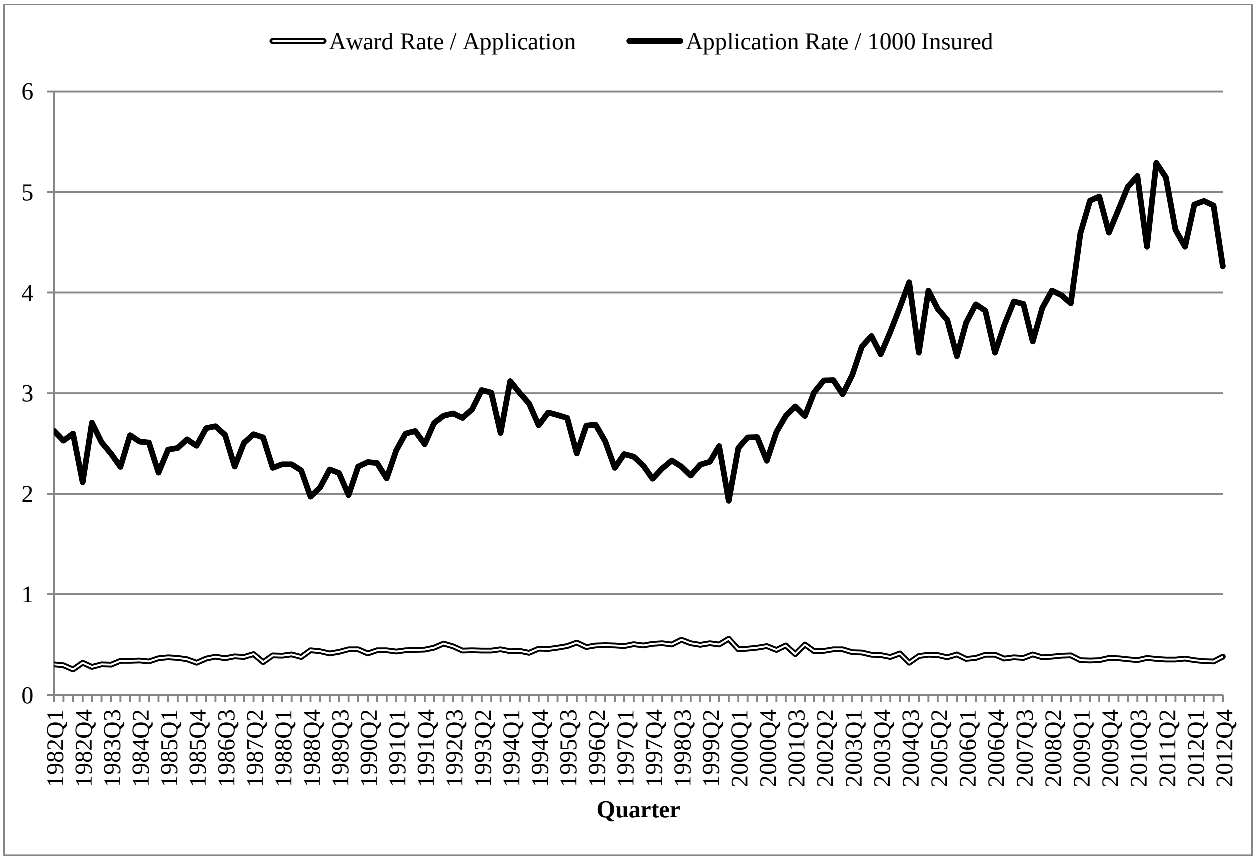
	Q1 1984–Q4 2012		**Q1 2008–Q4 2012**	
	MAPE	MAD	MAPE	MAD
ARIMAX	3.42%	0.10	2.85%	0.13
SAMA	4.50%	0.14	4.54%	0.21

Once again, the ARIMAX model clearly outperforms the SAMA model based on its lower MAPEs and MADs for both time periods. In comparison with the SAMA model, the ARIMAX model's MAPEs improve by 24 percent and 37 percent, and its MADs improve by 29 percent and 38 percent.

1.5 The Dependent Variable

While disability insurance award rates (i.e., approval rates) are somewhat influenced by the previously mentioned 16 factors, they are also determined by forces internal to the insurance provider such as organizational goals, strategies, policies and practices created and administered from within. This tends to reduce the volatility in approval rates and makes them more predictable than application rates. Not surprisingly, as seen in Figure 3, the application rates for SSDI among insured workers have exhibited much more variability than the acceptance rates among SSDI applicants. During the 31-year period of this study (1982–2012), the quarterly application rate per 1,000 insured workers ranged from a low of 1.929 in Q4 1999 to a high of 5.292 in Q1 2011, a rise of almost 274 percent. Over the 124 quarters in the data set, the application rate mean was 3.046 and the standard deviation was 0.869, yielding a coefficient of variation ($\hat{\sigma}/\hat{\mu}$) of 0.285. During the same period, the quarterly award rate, which is the proportion of applications approved, was relatively flat, ranging from 0.255 to 0.558, with a mean of 0.411 and a standard deviation of 0.063, yielding a considerably smaller coefficient of variation of 0.153. (Note that the coefficient of variation for application rates is 86.3 percent larger than that for award rates.) Figure 3 makes clear the contrast in the long-term slope and the volatility of the two time series.

Figure 3. Application and award rates for social security disability benefits.



This study focuses on modeling the more highly volatile, publicly available quarterly SSDI-application-rate/1000 insured time series presented in Figure 3. It serves well as a surrogate for private-sector submitted LTD claims experience in building time-series forecasting models. While the application-rate time-series patterns in the private sector are not created by all of the same forces that drive public-sector demand for disability payments, there are certainly many overlaps. In both settings, regional and national economic conditions heavily influence the rate of applications as do medical advancements and breakthroughs in the treatment of specific disorders. Other common influences include demographic shifts (e.g., aging baby boomers), technological improvements that can enhance one's ability to work and level of participation of females in the workforce.

2. CONSTRUCTION AND VALIDATION OF ARIMA AND ARIMAX MODELS

Section 2.1, Construction and Validation of an ARIMA Model, focuses on explaining and illustrating the steps in the methodology for constructing a pure ARIMA model. This illustration employs the previously introduced 124-point quarterly SSDI-application-rate time series (Q1 1982–Q4 2012). This discussion also includes all of the statistical assumptions that must be satisfied for an ARIMA model to be valid. Results from the analysis of residuals from final ARIMA and ARIMAX models are examined to ensure they meet the necessary conditions. Model-fitting results are then presented and evaluated using standard goodness-of-fit measures produced by the fitting process. In addition, the accuracy/precision of the holdout forecasts produced by the final pure ARIMA model are examined.

Section 2.2, Construction and Validation of an ARIMAX Model, is heavily patterned after Section 2.1, but focuses on explaining and illustrating the step-by-step methodology for building and validating an ARIMAX model. This discussion also includes an explanation of the much-expanded series of statistical assumptions that must be satisfied for an ARIMAX model to be valid. In keeping with the ARIMA discussion in Section 2.1, results from the analysis of residuals are reviewed and the quality of the ARIMAX model is evaluated in the context of both in-sample fitting and holdout-sample forecasting. Lastly, both sets of goodness-of-fit statistics are compared with their counterparts produced by the pure ARIMA model to assess the incremental explanatory value contributed by the exogenous variables.

2.1 Construction and Validation of an ARIMA Model

The AR (autoregressive) in ARIMA refers to previous (i.e., lagged) values of the dependent-variable time series. MA (moving average) refers to lagged error terms (i.e., residuals) created by the ARIMA model's inability to produce perfectly accurate estimates. So, ARMA (ARIMA without I) models are similar in appearance to a regression model with all of the right-hand-side (RHS) variables being lagged versions of the dependent variable y_t and lagged versions of the error term ϵ_t.

A general order ARMA (p, q) model with p autogressive terms (y_t 's) and q moving average terms (ϵ_t's) would be represented as

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \epsilon_t - \theta_1 \epsilon_{t-1} - \theta_2 \epsilon_{t-2} - \dots - \theta_q \epsilon_{t-q.}$$ [6]

In terms of structure, ARIMA (p, d, q) models are the same as ARMA (p, q) models where the time series has first been transformed by differencing. The d specifies the order of the differencing. For example, in Figure 5, the original undifferenced ($d = 0$) quarterly time series and the differenced once ($d = 1$) time series are graphed. In Figure 6, the original, undifferenced time series is differenced once ($d = 1$) by four, and then these differences are differenced again by one ($d = 2$). Since the time series must be stationary before it can be modeled with AR and MA terms,[7] differencing is commonly used to transform a nonstationary time series into a stationary time series where the mean and variance are statistically judged to be constant.

For example, a repeating daily time series that was strongly influenced by the day of the week (Sunday–Saturday) might likely be differenced by seven to remove the day-of-week effect. The

[6] Montgomery, Jennings, and Kulahci, *Introduction to Time Series*, 253.
[7] SAS Institute Inc. *SAS/ETS 9.2*, 210.

resulting differenced time series would then represent the week-to-week change in the daily data and the variance created by the day-of-week effect would be largely removed. At the same time, if there were no underlying weekly trend in the original time series, then these transformed data would likely appear to be stationary with a mean close to zero. However, if this same original (untransformed) time series were increasingly trending up in a quadratic fashion, then the differenced-by-seven time series would exhibit a positive linear trend (without the day-of-week influence), and the mean would not be constant over time. To address this lack of stationarity, differencing the resulting time series by one would remove the upward trend and cause the mean of the twice-transformed time series to be relatively constant. If both the mean and variance were indeed constant, the doubly differenced series would be stationary. Conveniently, the degree of stationarity of the transformed time series may be statistically evaluated using the augmented Dickey-Fuller test.[8]

Once a time series is statistically judged to be stationary, ARMA/ARIMA model building may begin. Identification of AR and MA terms requires the model builder to examine the autocorrelation coefficient function (ACF) and the partial autocorrelation coefficient function (PACF), to gain insights into the nature of the serial correlation.[9]

At the most basic level, there are two types of ARMA/ARIMA models, subset (i.e., additive)[10] and order. An order ARMA (p,q) model to estimate y_t is comprised of p terms involving $y_{t-1},\ y_{t-2}, \dots, y_{t-p}$ and q terms involving $\epsilon_{t-1},\ \epsilon_{t-2}, \dots, \epsilon_{t-q}$. In other words, the autoregressive terms would include lags of 1 through p and the moving-average terms would

[8] Ibid., 246.
[9] Nau, "Identifying the Numbers."
[10] SAS Institute Inc. *SAS/ETS 9.2,* 212.

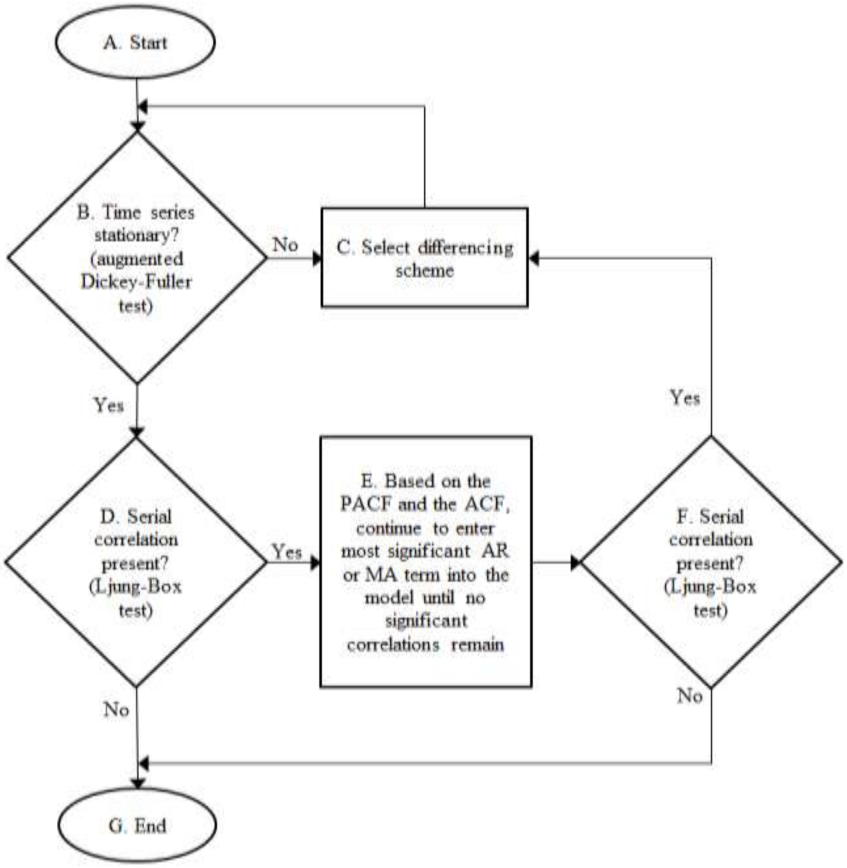
include lags 1 through q. In contrast, a subset or additive model includes only specified lags for the autoregressive terms and specified lags for the moving-average terms. Stige et al. states that, “Subset ARIMA models are often used to obtain parsimonious models that may be more interpretable than nonsubset ARIMA models,” and cites three other references that discuss their success in applying subset models.[11] The subset model-building approach was chosen for this effort for these reasons and because it facilitated the identification of models with much more finely tuned specifications, thus providing more attractive models from which to choose.

Identifying the form of an ARMA or ARIMA model is an iterative process that requires selecting appropriate differencing schemes to achieve stationarity as signaled by the augmented Dickey-Fuller test. Then, appropriately lagged AR and MA terms are introduced based on the significant patterns exhibited by their correlation functions. After the introduction of each AR or MA term, the residuals are re-examined for significance using the ACF and PACF. The process continues until these two correlation functions provide no further statistical clues to indicate that any AR or MA terms are missing. At this point, the Ljung-Box test for white noise[12] may be used to statistically evaluate the degree to which the residuals are free of serial correlation. The statistical details of this are discussed in Montgomery, Jennings and Kulahci.[13]

The flowchart in Figure 4 captures the sequence of steps that must be followed to construct a valid pure ARIMA model. As indicated by the two nested looping structures (B→C→B and B→D→E→F→C→B), this process may take many iterations to complete.

[11] Stige, et al., “Thousand-Year-Long Chinese Time Series.”
[12] SAS Institute Inc. *SAS/ETS 9.2,* 194.
[13] Montgomery, Jennings, and Kulahci, *Introduction to Time Series*, 57.

Figure 4. ARIMA model-building algorithm


A. Start
B. Time series stationary? (augmented Dickey-Fuller test)
No
C. Select differencing scheme
Yes
Yes
D. Serial correlation present? (Ljung-Box test)
Yes
E. Based on the PACF and the ACF, continue to enter most significant AR or MA term into the model until no significant correlations remain
F. Serial correlation present? (Ljung-Box test)
No
No
G. End

Building an ARIMA model for the 124-quarter SSDI-application-rate time series requires executing the five steps (labeled B, C, D, E and F in the flowchart above) at least once.

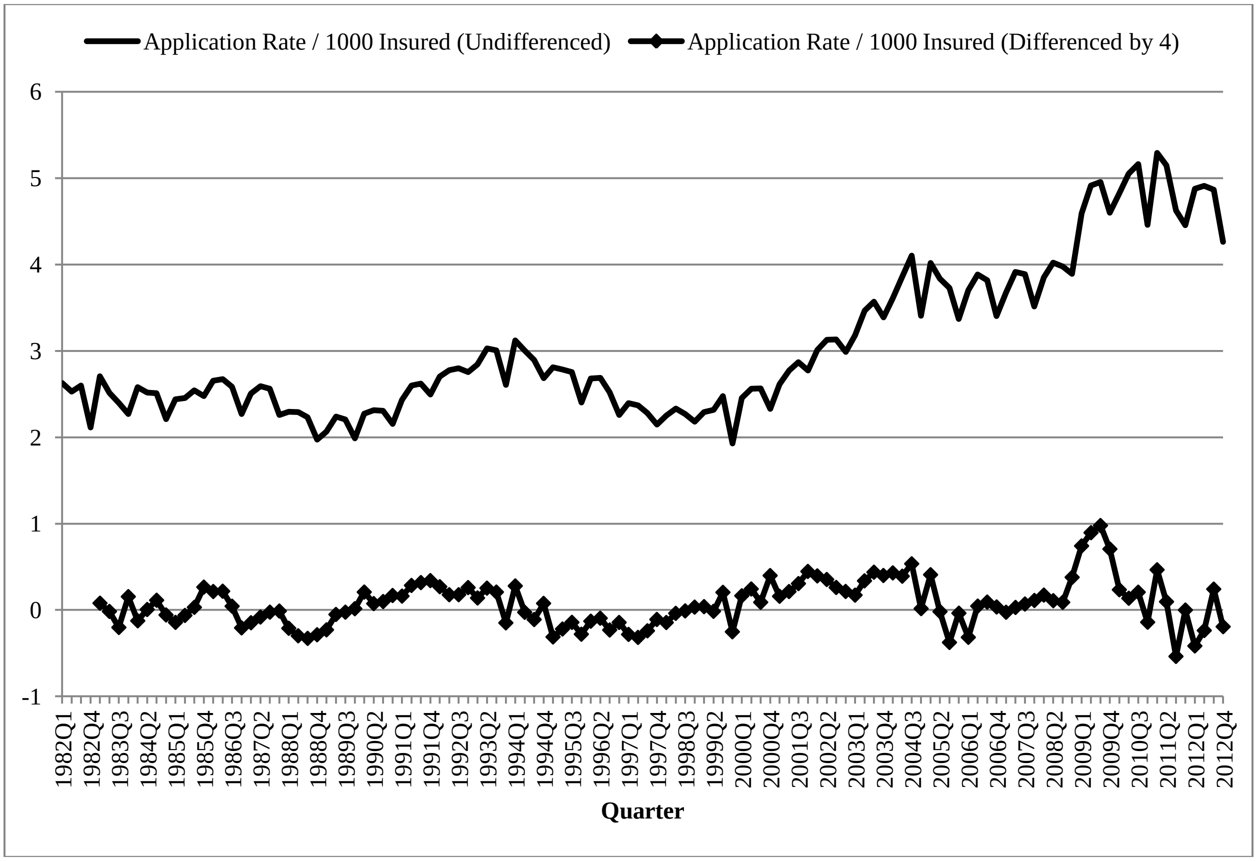
1. The raw (undifferenced) time series must be evaluated for stationarity using the augmented Dickey-Fuller test (Step B) and transformed, if necessary. The SAS output in Chart 2 shows that, in its raw form, the time series is not stationary. The *p*-values for lags 0–4 are very large (0.5685–0.9944) and do not support rejection of the null hypothesis that the series is not stationary. (Note: The single mean test that examines the null hypothesis that the time series has a constant mean is appropriate, as the time series must meet this condition before it can be modeled with the ARIMA methodology.)

Chart 2. SAS output: Augmented Dickey-Fuller test.

```
Type           Lags    Pr < Tau

Single Mean       0     0.5685
                  1     0.8170
                  2     0.9469
                  3     0.9944
                  4     0.9257
```

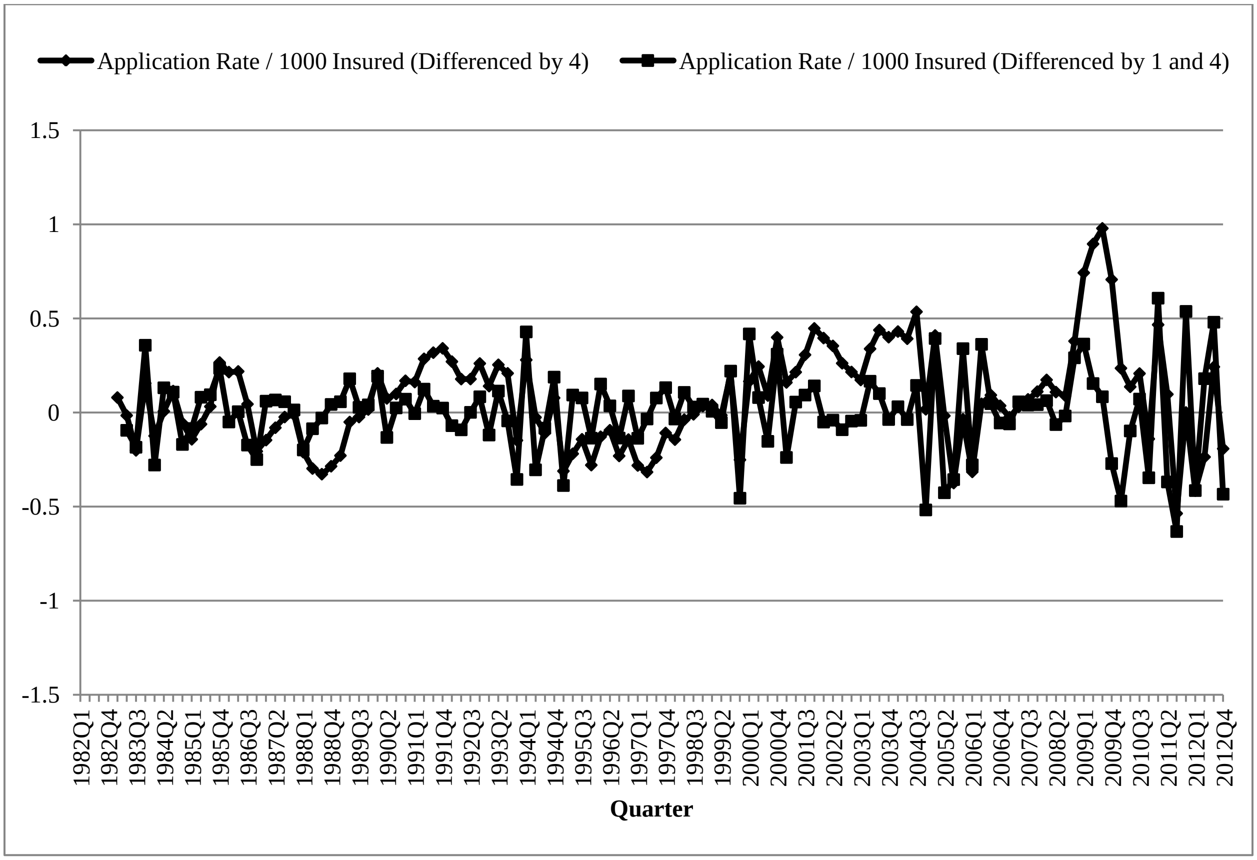
2. In Figure 5, an upward trend in the undifferenced SSDI data is very apparent, as is the four-period seasonality made evident by the behavior of Q4, which is the smallest quarter for each of the 31 years. To remove the seasonality, differencing this quarterly time series by 4 produces a much more stable pattern (Step C). Figure 5 shows that the variance in the time series is significantly reduced by this transformation. Centered around 0, the differenced time series ranges from −0.5 to 1.0, in contrast with the original series that ranges from under 1.9 to over 5.2. Thus, differencing has reduced the sample range from 3.3 to 1.5 (in approximate terms).

Figure 5. A comparison of the raw SSDI application rate data and the differenced data.



3. While the differenced-by-four time series appears to be much more stable, it still exhibits a slight upward trend. This observation implies that further differencing by one might prove productive. Figure 6 contrasts the twice-differenced time series (first by 4 and then by 1) with the series that results from the single differencing by 4.

Figure 6. A comparison of the SSDI application rate data using two different differencing schemes.



From purely a visual assessment, it appears that the best differencing transformation for providing a constant mean and minimum variance has been found. (Note: A commonly used rule of thumb[14] is that the optimal order of differencing produces the lowest standard deviation.) This is further substantiated by examining the five highly significant (<0.0001) p-values for lags 0–4 in the augmented Dickey-Fuller test (Step B) presented in the single-mean portion of the SAS output shown in Chart 4. The five 0^+ p-values for the single-mean model with lags 0–4 support rejection of the five null hypotheses asserting a unique mean at each lag value (0–4). This implies that the differenced data are

[14] Nau, “Identifying the Order.”

stationary and that there is a single mean. Lastly, while the (1,4) differencing scheme made the transformed series stationary, the p-values from the autocorrelation check for white noise remained very small and suggested that significant AR and/or MA terms were needed (Step D) to remove the highly significant autocorrelation still present in the twice-differenced ($d = 2$) series.

Chart 3. SAS output: Autocorrelation check for white noise.

```
 To      Chi-          Pr >
Lag    Square   DF   ChiSq  -------------------Autocorrelations------------------

  6     68.39    6  <.0001   -0.382    -0.019     0.298    -0.514    0.173   0.144
 12     72.19   12  <.0001   -0.084     0.043    -0.034    -0.126    0.052   0.013
 18     73.53   18  <.0001    0.038    -0.037    -0.073    -0.035   -0.018  -0.009
 24     94.93   24  <.0001    0.085    -0.043    -0.072     0.191   -0.213   0.217
```

Chart 4. SAS output: Augmented Dickey-Fuller test.

```
Type           Lags   Pr < Tau

Single Mean       0    <.0001
                  1    <.0001
                  2    <.0001
                  3    <.0001
                  4    <.0001
```

4. It is worth noting that 14 other differencing schemes and natural log transformations of differencing schemes were examined. In addition to diff (1,4), diff (1,2) also produced five very attractive p-values (<.0001) to support the assumption of stationarity. However, the standard deviation of the diff (1,4) time series was 0.221, about half of the larger standard deviation estimate of 0.407 produced by diff (1,2).

5. With a stationary time series in hand, ARIMA model building began. Examination of the ACF plot of residuals (Step E) indicated that an MA4 term was needed based on its highly

significant negative correlation (−0.51367). The PACF provided evidence that an AR4 term with a correlation of −0.38350 was also significant, but not as significant as the MA4, so the MA4 was introduced as the first term in the ARIMA model. This MA4 term had a coefficient of 0.83511 and a highly significant t-statistic of 12.13, and produced a model with a Schwarz Bayesian criterion (SBC)[15] of −83.076.

6. Next, the ACF and PACF (Step E) for the first-stage residuals were re-examined to identify further MA or AR candidates that had the potential to make a statistically significant explanatory contribution to modeling the stage 1 residuals. The ACF at lag one and the PACF at lag one both had highly significant identical correlations of (−0.29184). However, the approximations for the standard errors for the PACF and the ACF used in computing the 95 percent confidence interval are slightly different. For the PACF, the standard error is approximately $\sqrt{\frac{1}{n}}$, where n is the number of data points in the time series under scrutiny.[16] (Note: After differencing by 1 and 4, the original time series with 124 quarters of data was reduced to a doubly differenced time series of p = 119 observations.) For the ACF, the standard error is approximately $\sqrt{\frac{(1+2\sum_{q=1}^{k-1} r_q^2\,)}{n}}$, where k is the lag of the ACF being examined and r_q is the autocorrelation at lag q,[17] which reduces to $\sqrt{\frac{1}{n}}$ for k = 1. To break this tie, the SBCs for the (AR1, MA4) model and the (MA1, MA4) model were compared, and, as shown in Chart 5, the (AR1, MA4) model prevailed with a preferred SBC of −89.324. So, the second term to enter was the AR1, as discussed above and shown in Row

[15] Beal, “Information Criteria Methods.”
[16] SAS Institute Inc. *SAS/ETS 9.2,* 240.
[17] Pecar, “Association Between.”

2 of the Chart 5. Note that the selection of the AR1 term over the MA1 term had a substantial impact on the quality of the model. While the PACF(1) spike and the ACF(1) spike were identical in terms of their significance levels, the positive impact of adding the AR1 to the initial MA4 term was clearly much greater, as reflected by the magnitude of the t-statistics for their coefficients (i.e., 3.38 vs. 1.93, respectively). Lastly, the t-statistics for the MA4 term in the two models are very different (i.e., 11.39 in Row 2 vs. 5.31 in Row 3), and reflect the enhancing role of the AR1 term and the detracting role of the MA1 on the MA4-foundation term common to both models. Further, the introduction of the AR1 term only decreased the MA4 t-statistic by 1.22, from 12.61 to 11.39. In contrast, adding the MA1 to the MA4-foundation model reduced the MA4 t-statistic by 7.30, from 12.61 to 5.31. This is not surprising because the lag 4 error terms were bound to capture much of the explanatory value that the lag 1 error terms would have captured in isolation. (Note: The correlation between MA4 and MA1 was 0.428, while the [AR1, MA4] correlation was about two times less at 0.196.)

Chart 5. ARIMA model-building results.

Row #	Model	SBC	Term	Coefficient	t	Pr > |t|
1	MA4	−84.027	MA4	0.84565	12.61	<.0001
2	AR1, MA4	−89.324	AR1	−0.30254	−3.38	0.0007
			MA4	0.80895	11.39	<0.0001
3	MA1, MA4	−86.402	MA1	0.28010	1.93	0.0536
			MA4	0.71989	5.31	<0.0001

7. A two-stage re-examination of the ACF and PACF (Step E) of the (AR1, MA4) model’s residuals suggested that first an AR10 term, and then an AR3 term should be added.

Further examination of the ACF and PACF of the (AR1, AR3, AR10, MA4) model's residuals provided no evidence that further AR or MA terms were needed. The preliminary subset ARIMA model is shown in Chart 6 as are the t-statistics and the p-values that demonstrate the strong explanatory power of the MA4, AR1, AR3 and AR10 terms.

Chart 6. SAS output: Final ARIMA model.

```
                                 Standard                 Approx
Parameter      Estimate             Error    t Value    Pr > |t|     Lag

MA1,1           0.78139           0.07645      10.22      <.0001       4
AR1,1          -0.31102           0.08355      -3.72      0.0002       1
AR1,2           0.24543           0.08396       2.92      0.0035       3
AR1,3          -0.27565           0.09699      -2.84      0.0045      10
```

8. The autocorrelation check of residuals (Step F) over the range of lags from 1–24, provided little evidence that autocorrelation remained in the residuals. With four p-values that ranged from 0.0919 to 0.4340, as shown in Chart 7, there was not sufficient evidence to support rejection of the null hypotheses that the residuals were white noise. Thus, the (AR1, AR3, AR10, MA4) model appeared to be sound, and the search process concluded (Step G).

Chart 7. SAS output: Autocorrelation check of residuals.

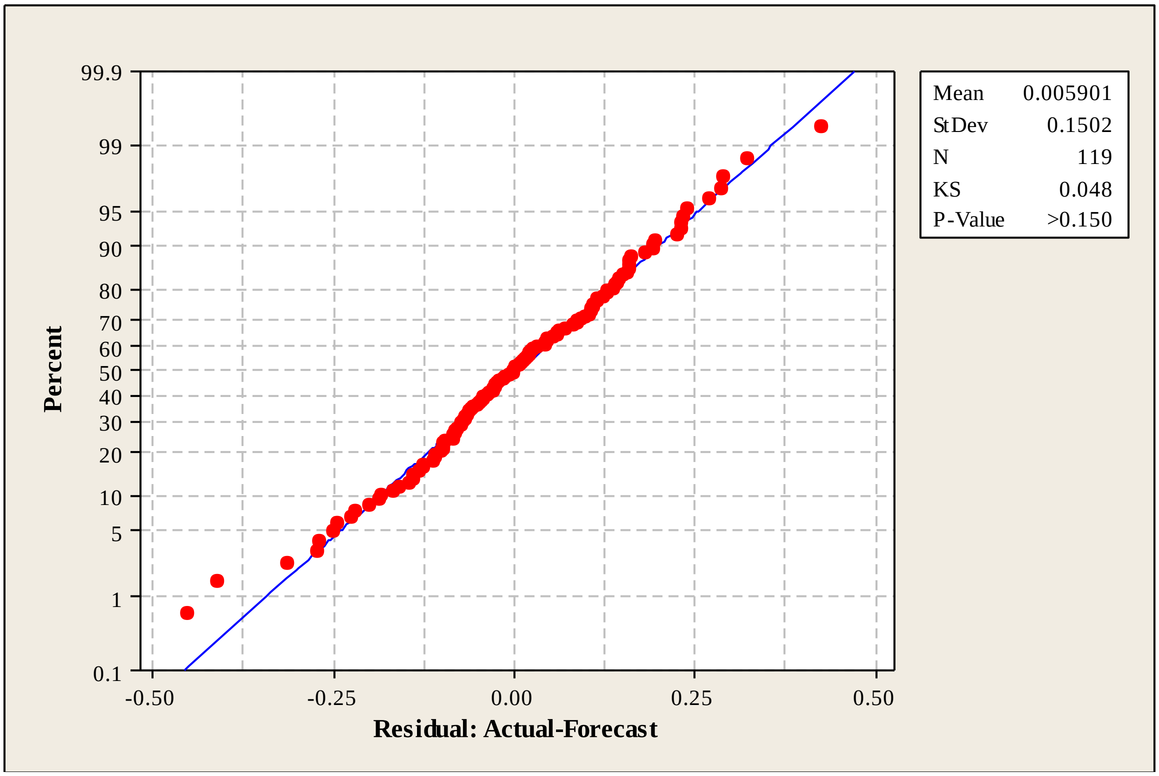
```
To    Chi-          Pr >
Lag   Square   DF   ChiSq   --------------------Autocorrelations--------------------

  6    1.93     2   0.3802   -0.010    0.030    0.006   -0.024    0.015    0.116
 12    7.99     8   0.4340    0.095    0.008   -0.147   -0.020   -0.039    0.116
 18   17.24    14   0.2436   -0.061   -0.155   -0.155   -0.096   -0.076    0.001
 24   28.79    20   0.0919   -0.006    0.064   -0.174    0.085   -0.057    0.180
```

9. In addition to performing the diagnostic Ljung-Box test to check for independence of the residuals, there are two other assumptions relating to residuals that must be validated: normality and homoscedasticity.[18]

    - Normality. The residuals should be normally distributed so that the t-statistics used to assess the significance of AR and MA terms are valid. A test often used for this purpose is the Kolmogorov-Smirnov (K-S) test,[19] which examines goodness of fit and the maximum difference between the observed cumulative distribution function (CDF) and a fully specified hypothesized cumulative distribution. As the vertical distance between the two CDFs increases, the K-S statistic also increases, which discourages acceptance of the null hypothesis of normally distributed errors. In practice, the mean and standard deviation of the hypothesized normal cumulative distribution function are often estimated from the sample data set,[20] resulting in conservatively approximated, rather than exact, p-values.

[18] Yurekli and Kurunc, "Testing the Residuals."
[19] National Institute of Standards and Technology, "Kolmogorov-Smirnov."
[20] SAS Institute Inc. "Tests for Normality."

Figure 7. Normal probability plot of residuals.



In the Minitab-generated Figure 7 constructed with the 119 residuals from the (AR1, AR3, AR10, MA4) fitted model, the K-S statistic is 0.048 with an attractive estimated p-value of >0.150 for the N(0.005901, 0.1502) fitted distribution. This does not support rejection of the null hypothesis that the (AR1, AR3, AR10, MA4) model residuals are normally distributed. The average of the 119 residuals$(\bar{r})$ is close to zero (0.005901) and the standard error (S.E.) = StDev/$\sqrt{n}$ = 0.1502/$\sqrt{119}$ = 0.0138. As such, the mean of the residuals is not statistically different from zero since Z = $[\bar{r} - E(r)]$/S.E. = [0.005901 – 0]/0.0138 = 0.427.

- Homoscedasticity. Residuals from an ARIMA model should display a constant variance in order to support proper calculation of the unbiased standard errors that are part of the t- and F-statistics needed for hypothesis testing. Biased standard errors can lead to improper rejection of null hypotheses asserting the statistical significance of AR and/or MA terms and, thus, the composition of the ARIMA model. White's test[21] attempts to establish whether or not the variance is changing. The White's test SAS output shown in Chart 8 indicates that the (AR1, AR3, AR10, MA4) model residuals do not exhibit heteroscedasticity at the $p =$ 0.0500 level. (Note: this very marginal p-value suggests there may be environmental influences not captured by this pure ARIMA model and that ARIMAX modeling with exogenous variables may prove useful.)

Chart 8. SAS output: White's test.

```
DF    Chi-Square    Pr > ChiSq

 2          5.98        0.0503
```

Having satisfied the assumptions of independence, normality and unchanging variance of the ARIMA residuals, the integrity of the (AR1, AR3, AR10, MA4) model has been established.

10. In addition to the SBC (valued at −94.537), other goodness-of-fit measures tabulated for the winning (AR1, AR3, AR10, MA4) model were rolling four-quarter MADs and rolling four-quarter MAPEs calculated over the fitted data and recalculated over the holdout-

[21] SAS Institute Inc. "Heteroscedasticity."

forecast data. In both cases, the first estimate included in the calculation of the rolling statistics was for period 25 (Q1 1988). As a consequence, the fit statistics for both the rolling MAPE and rolling MAD were computed over the 100 most recent four-quarter periods beginning at Q1 1988 and ending at Q4 2012.

In estimating the coefficients for the (AR1, AR3, AR10, MA4) model and in producing the rolling four-quarter holdout-forecasts, all of the historical data prior to the first forecasted quarter (Q1 1988) were employed. As such, the tabulated results realistically reflect the quality of the projections that would have been made at the time. In contrast, the four-quarter fit model is static because its parameters were estimated only once using all of the same 119 historical data points. Its rolling MADs and MAPEs reflect the extent to which one static model fit 100 subsequent four-quarter periods from Q1 1988 to Q4 2012.

Chart 9. ARIMA four-quarter rolling MAPEs and MADs (Q1 1988–Q4 2012).

Estimate Type	Goodness-of-Fit Measures	n	Mean	Std. Dev.	Min.	Max.
Fit	MAPE	100	3.72	1.51	1.18	9.14
	MAD	100	0.12	0.06	0.04	0.31
Forecast	MAPE	100	5.17	2.70	1.18	13.74
	MAD	100	0.16	0.10	0.04	0.63

Examining the mean, standard deviation, minimum and maximum for the 100 four-quarter rolling MAPEs and 100 four-quarter rolling MADs for the fit estimates and the forecast estimates, shown in Chart 9, the pure ARIMA model seems to be quite accurate. Naturally, the mean and standard deviation goodness-of-fit measures for the fits dominate their counterparts from the holdout forecasts. In both cases, however, the mean four-quarter rolling MADs of 0.12 and 0.16 are quite respectable in relation to the range of the raw time-series values (1.93–5.29) over the 25-year period. The rolling MAPEs averaged 3.72 percent for the fit model and 5.17 percent for the forecast model over the same 25 years.

All of these goodness-of-fit statistics are quite attractive given the substantial expanse of the time period, as well as the dynamics of the economy and the evolution of SSA's policies/persuasions. After all, this forecasting methodology relies only on historical time-series patterns to make its projections, and has no mechanisms for directly integrating the impacts of exogenous influences. Addressing this deficit is the topic of Section 2.2.

2.2 Construction and Validation of an ARIMAX Model

ARIMAX is an acronym for autoregressive integrated moving-average with exogenous variables. It is a logical extension of pure ARIMA modeling that incorporates independent variables which add explanatory value. Conceptually, it is a merging of regression and ARIMA modeling.[22] When the AR and MA terms in a pure ARIMA model are not sufficient to provide an acceptably high R^2 (or some other measure of a model's overall explanatory power), it is only natural to look for other driving phenomena whose influence over time is not sufficiently embedded in the historical values of the dependent time series.

Building an ARIMAX model calls for combining the predictive value of both the trailing time-series values themselves (y_t) and the trailing model errors (ϵ_t) with the predictive value of exogenous variables. As a simple example, if a set of exogenous variables serving as independent variables in a multiple regression were all properly signed and highly significant, did not exhibit significant cross-correlation and produced a high R^2 with the time series of residuals approximating white noise, there would be no need for ARIMAX modeling. However, if that same multivariate regression equation generated residuals that exhibited significant serial correlation, then pure ARIMA modeling of the residuals would be required in order to remove the serial correlation so that t-statistics could be properly calculated and the significance of the independent variables could be properly judged.

The ARIMAX approach to time-series model building has two phases. This methodology traditionally begins with a logically attractive and statistically sound regression model. Then, the

[22] SAS Institute Inc. *SAS/ETS 9.2,* 21.3

errors from the regression are modeled with AR and MA terms to remove any statistically significant serial correlation that remains in the residual time series. (Note: The final ARIMAX model is composed of exogenous variables along with AR and/or MA terms, so it is sometimes useful to conduct an exploration for exogenous terms using the residuals from a pure ARIMA and then look at their cross-correlations with other explanatory variables.[23] This is particularly true if the pure ARIMA is stable and it explains the vast majority of the variation of the dependent variable. After all, it is the exogenous, AR and MA terms that collectively comprise the final model, and they need to complement each other to maximize the explanatory power of the RHS variables while eliminating any significant autocorrelation among the residuals.)

While the traditional (regression-first) two-phase process appears to be straightforward, it is not. There is a powerful interaction created by the integration of AR and MA terms into a multiple regression model that frequently creates the need for an iterative search process. This is particularly true if the pool of exogenous-variable candidates is large. For example, when a new exogenous variable is selected in a stepwise process and introduced to the ARIMAX model, it may well disrupt the white-noise pattern of the residuals from the previous step. This concern would need to be addressed with the addition of new AR and/or MA terms to re-establish the random pattern of residuals. In turn, the newly added AR and/or MA terms may explain variation previously explained by one or more resident exogenous variables, which then forces one or more of these impacted independent variables out of the ARIMAX model. This disruption, in turn, produces new residuals whose ACF and PACF must be examined to determine if additional AR or MA terms should be added to the model. Once the serial correlation is removed, additional

[23] Nau, “ARIMA Models.”

exogenous variables may need to be removed due to lack of significance, and so the cycle continues.

There are six statistical assumptions that must be examined/re-examined to ensure that the resulting ARIMAX model is valid at each stage of its evolution. Two of these six assumptions (denoted as assumptions 1 and 2) pertain to the residuals produced by the model, and the other four (denoted as assumptions 3–6) relate to the exogenous variables that comprise the model.

- Assumption 1. As discussed in Section 2.1, ARIMA model building may not commence until the time series is stationary.[24] This requires that the mean and the variance of the residual series are unchanging over time. The degree of stationarity of the residuals may be statistically evaluated using the augmented Dickey-Fuller test.[25] As before with pure ARIMA model building, the p-values for the augmented Dickey-Fuller test for a single mean must be acceptably small to ensure stationarity. If the residuals produced by the regression are not sufficiently stationary, the level of stationarity may oftentimes be improved by applying the same well-chosen differencing scheme (or another transformation) to the dependent and to all of the exogenous variables.

- Assumption 2. In addition to stationarity, the residual series must not exhibit significant serial correlation (i.e., autocorrelation). The Ljung-Box test may be used to statistically evaluate the degree to which the residuals are serially correlated. If significant serial correlation exists among the residuals, it may be reduced by adding an appropriate

[24] SAS Institute Inc. *SAS/ETS 9.2,* 215.
[25] Ibid.,158.

combination of one or more significant AR and/or MA terms identified from the PACF and ACF, respectively.

- Assumption 3. The estimated coefficient for an exogenous variable must be significantly different than 0, as judged by its t-statistic. However, the calculation of the significance levels of t-statistics (p-values) for regression coefficients assumes that the regression residuals are white noise. If Assumption 2 is violated, and these residuals are not white noise, then their serial correlation must be removed with ARIMA modeling. This calls for the pure ARIMA modeling process discussed in Section 2.1 and outlined in Figure 4.

- Assumption 4. An exogenous variable must not display evidence of receiving feedback from the dependent variable. That is, an attractive exogenous-variable candidate should display a significant causal relationship with the dependent variable without the dependent variable displaying a causal relationship with it. The directions of significant causality between an exogenous variable and the dependent variable may be tested using the Granger causality test[26]. If reverse causality is detected, the exogenous variable must be removed from the pool of independent-variable candidates. This test must be performed on the dependent and independent variable in their current form (untransformed or transformed).

- Assumption 5. The sign of the coefficient for each significant exogenous variable must be reasonable. The expected (i.e., reasonable) sign can be determined prior to model building by examining the signs of exogenous-variable correlation-coefficients that display a significant

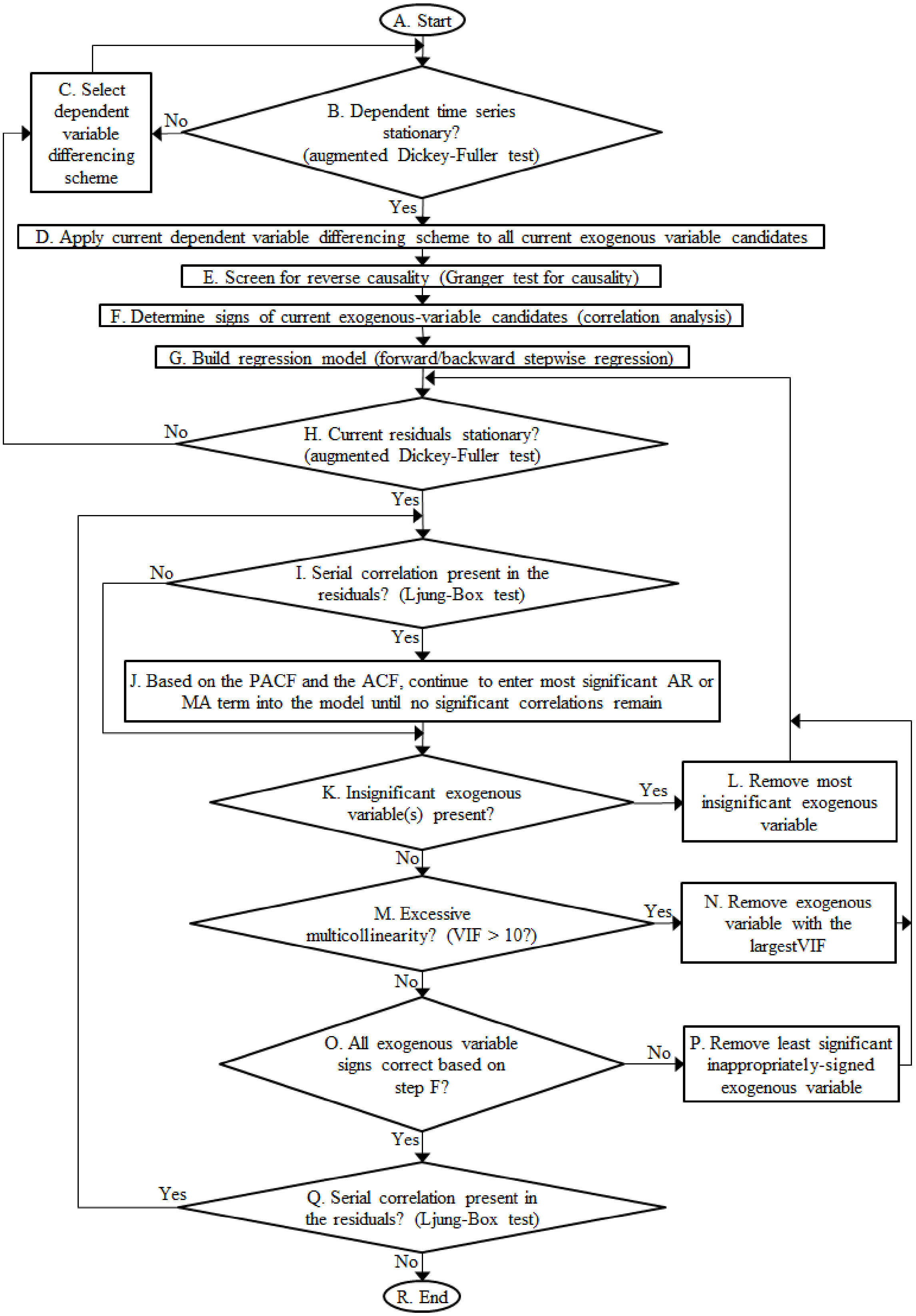
[26] SAS Institute Inc. “Bivariate Granger.”

correlation with the dependent variable. If the dependent variable required a transformation to achieve stationarity, that same transformation would also be applied to the independent-variable candidates, and the bivariate-correlation analysis would then focus on the pair of transformed variables.

- Assumption 6. The surviving exogenous variables comprising the final model must not exhibit a significant degree of multicollinearity. To meet this condition, one at a time, each of the surviving exogenous variables must be individually tested for significant multicollinearity using the variance inflation factor (VIF = $1/[1 - R^2]$) to ensure they are all sufficiently linearly independent. When the multicollinearity among exogenous variables is too strong, least squares estimation becomes inefficient, causing the standard errors of the estimates to become large and result in overstated p-values. A VIF of 10 or less is generally considered to indicate an acceptable level of correlation among the exogenous variables. The VIF calculations must be performed for each of the independent variables expressed in their current form (i.e., transformed or untransformed). (Note: Each of the k R^2s is actually calculated by selecting one of the k independent variables to assume the role of dependent variable in a regression with all of the remaining $[k - 1]$ independent variables serving as independent variables.) Thus, the VIF ≤ 10 rule of thumb is the equivalent of requiring that each independent variable's variation be no more than 90 percent explainable based on the weighted aggregate of the other $(k - 1)$ independent variables.

The flowchart shown in Figure 8 presents the algorithm used to build a valid ARIMAX model. It is constructed using an iterative scheme based largely on the principles embodied in the six assumptions above. As indicated by the maze of 40 nested and unnested looping structures, the examination/re-examination of assumptions 1–6 provides the foundation for the model-building methodology. Excluding the A. Start and the R. Stop nodes, there are 16 steps, many of which are executed numerous times. The five core steps (B–F) in the ARIMA model-building algorithm presented earlier are also embedded in this flowchart. The substantial increase in steps from 5 to 16 is largely based on the complexities introduced by marrying the elements/requirements of regression model building with those of the pure-ARIMA model building.

Figure 8. ARIMAX model-building algorithm.


A. Start
B. Dependent time series stationary? (augmented Dickey-Fuller test)
No
C. Select dependent variable differencing scheme
Yes
D. Apply current dependent variable differencing scheme to all current exogenous variable candidates
E. Screen for reverse causality (Granger test for causality)
F. Determine signs of current exogenous-variable candidates (correlation analysis)
G. Build regression model (forward/backward stepwise regression)
H. Current residuals stationary? (augmented Dickey-Fuller test)
No
Yes
I. Serial correlation present in the residuals? (Ljung-Box test)
No
Yes
J. Based on the PACF and the ACF, continue to enter most significant AR or MA term into the model until no significant correlations remain
K. Insignificant exogenous variable(s) present?
Yes
L. Remove most insignificant exogenous variable
No
M. Excessive multicollinearity? (VIF > 10?)
Yes
N. Remove exogenous variable with the largestVIF
No
O. All exogenous variable signs correct based on step F?
No
P. Remove least significant inappropriately-signed exogenous variable
Yes
Q. Serial correlation present in the residuals? (Ljung-Box test)
Yes
No
R. End

There are three families of looping structures labeled B, H and I, with two B-loops, six H-loops and two I-loops. Note that the six H-loops are nested inside the B2 loop and the two I-loops both are nested inside all six H-loops. This creates 40 loops in total: 10 unnested, 18 singly nested and 12 doubly nested.

- B-loops:
  1. B→C→B
  2. B→D→ E →F→ G→ **H**→ C→ B
- H-loops:
  1. H→ **I**→ K→ L→ H
  2. H→ **I**→ J→ K→ L→ H
  3. H→ **I**→ K→ M→ N→ H
  4. H →**I**→ J→ K→ M→ N→ H
  5. H →**I**→ K→ M→ O→ P→ H
  6. H →**I**→ J→ K→ M→ O→ P→ H
- I-loops:
  1. I→ K→ M→ O→ Q→ I
  2. I→ J→ K→ M→ O→ Q→ I

One of the early tasks of ARIMAX model building is to identify and preliminarily evaluate the logical/statistical attractiveness of exogenous variable candidates. SSDI application rates have been well studied over the years, so for this study, there was no shortage of attractive explanatory-variable candidates. The U.S. Social Security Administration website has a wealth of literature on this topic, most of which relates to the condition of the U.S. economy. For this model-building exercise, 14 exogenous-variable candidates were identified. Expecting that these 14 exogenous variables might lead quarterly SSDI application rates, each of 14 candidates was assigned to lags of 0, 1, 2, 3 and 4 quarters, thus creating 70 independent-variable candidates in total. Chart 10 provides each variable's name, description and data source.

Chart 10. 70 Exogenous-variable candidates.

SAS Variable (Each With Lags 0–4)	Description	Source
total_permits	total single and multifamily permits	Moody's Analytics
housing_starts	housing starts (in millions)	Moody's Analytics
median_home_price	median single-family home price (in thousands)	Moody's Analytics
total_fixed_invest	total fixed investment (in billions of 2005 dollars)	Moody's Analytics
wse	number of nonfarm, payroll jobs in the U.S economy (in thousands)	bls.gov/data: Employment; Employment, Hours, and Earnings—National (Current Employment Statistics)
resident_employment	number employed (in thousands)	bls.gov/data: Employment; Labor Force Statistics (Current Population Survey)
unemployment	unemployment rate	bls.gov/data: Unemployment; Labor Force Statistics (Current Population Survey)
num_unemployed	number unemployed (in thousands)	bls.gov/data: Unemployment; Labor Force Statistics (Current Population Survey)
cpi_urban	Consumer Price Index for all urban consumers – all items	bls.gov/data: Inflation & Prices; All Urban Consumers (Consumer Price Index)
nom_gdp	GDP (in billions of current dollars)	bea.gov/national/index.htm: Current-dollar and "real" GDP
real_gdp	GDP (in billions of 2005 dollars)	bea.gov/national/index.htm: Current-dollar and "real" GDP
mean_earnings	mean individual weekly earnings	Moody's Analytics
weekly_hours	average number of weekly hours: total nonfarm	Moody's Analytics
hourly_earnings	average hourly earnings: total nonfarm	Moody's Analytics

Lagged (1–4) forms for exogenous variables are denoted with suffixes.

Building an ARIMAX model requires executing some combination of the 16 (or fewer) steps in Figure 8, the ARIMAX model-building flowchart.

1. As in the pure ARIMA model-building process discussed earlier, the first two steps involve testing the dependent time series for stationarity using the augmented Dickey-Fuller test (Step B) and, if required, selecting an appropriate differencing scheme for the dependent variable (Step C).

2. Frequently, for consistency, the differencing scheme chosen for the dependent variable during pure ARIMA model building can be applied to exogenous-variable candidates to make them stationary as well. With both the dependent and independent variables stationary, the correlations are more likely to be stable over time.[27] In this case, all of the exogenous-variable candidates became stationary after being differenced by 1 and 4. Note that these transformations must be performed at an early stage in the model-building process so that subsequent tests such as the Granger test of causality will employ the exogenous variables in the form they will subsequently appear in the final model.

3. Next, the transformed exogenous variables are screened using the Granger test for causality (Step E) to remove any variables that display significant evidence of reverse causality, as discussed above in Assumption 4. Any variable with a p-value below 0.0500 led to rejection of the null hypothesis of no reverse causality, thus eliminating it as a candidate for inclusion in the model. This reduced the pool of exogenous-variable

[27] Nau, "Identifying the Numbers."

candidates from 70 to 57. The 13 eliminated variables and their associated p-values are shown in Chart 11.

Chart 11. Exogenous-variable candidates eliminated by Granger test for causality.

Variable_Name	Chi-Square	Pr > ChiSq
wse_3	10.08	0.0015
wse_1	9.24	0.0024
mean_earnings_2	8.42	0.0037
total_fixed_invest_3	7.15	0.0075
unemployment_2	6.76	0.0093
real_gdp_2	6.42	0.0113
num_unemployed_2	6.41	0.0114
weekly_hours_2	6.40	0.0114
total_fixed_invest_1	6.37	0.0116
cpi_urban_1	6.19	0.0128
nom_gdp_2	5.73	0.0167
resident_employment	5.02	0.0251
median_home_price_1	4.79	0.0286

4. As previously discussed in Assumption 5, the “correct” signs for the remaining transformed exogenous-variable candidates (Step F) must be determined by performing an analysis of the correlations between the transformed exogenous-variable candidates and the transformed dependent variable. Variables that do not display a significant correlation (p < 0.0500) are not assigned an expected sign and are removed from the pool of independent-variable candidates. Chart 12 presents the 14 surviving exogenous-variable candidates, their correlation coefficients and their corresponding p-values. (Note: Since SAS employs a two-tailed test of significance for correlation coefficients, the p-value threshold of 0.1000 was employed.)

Chart 12. Exogenous-variable candidates remaining after correlation analysis.

Exogenous Variable	Corr. Coef.	p-value
wse	−0.31133	0.0007
total_fixed_invest	−0.27009	0.0035
cpi_urban	−0.25722	0.0055
real_gdp_1	−0.25392	0.0062
nom_gdp	−0.24711	0.0078
nom_gdp_1	−0.23703	0.0108
unemployment_1	0.22976	0.0135
num_unemployed_1	0.22806	0.0142
weekly_hours_1	−0.20689	0.0265
mean_earnings_1	−0.20241	0.0300
median_home_price	−0.17641	0.0593
num_unemployed	0.17640	0.0593
real_gdp	−0.17133	0.0671
cpi_urban_4	0.16875	0.0714

The signs of these 14 significant correlation coefficients are retained for subsequent use in Step O to eliminate from the ARIMAX model any of the significant exogenous variables whose coefficients are incorrectly signed.

5. The forward/backward stepwise regression procedure in SAS provides an iterative approach to regression model building that both adds significant variables to the model and removes variables from the model that become insignificant (Step G). The process begins by determining the exogenous-variable candidate with the smallest p-value that is less than the chosen “enter” significance-level threshold of 0.1000 and adding that variable to the model. Next, the p-values of all of the variables in the current model are checked, and the variable with the largest p-value above the chosen “stay” significance-level threshold of 0.0500 is removed. This process of adding and deleting exogenous

variables continues until there are no variables that meet either criterion.[28] The final results of the three iterations of the stepwise process are displayed in the SAS output of Chart 13.

Chart 13. SAS output: Summary of stepwise selection.

```
     Variable      Variable    Number   Partial   Model
Step Entered       Removed     Vars In  R-Square R-Square C(p)   F Value  Pr > F

  1  wse                           1      0.0968   0.0968  0.8865  12.22  0.0007
  2  num_unemployed                2      0.0238   0.1206 -0.1150   3.06  0.0830
  3                num_unemployed  1      0.0238   0.0968  0.8865   3.06  0.0830
```

As shown in the SAS output in Chart 14, the final iteration of the stepwise-regression process results in a standard regression model containing one exogenous variable, which is highly significant (with a p-value of 0.0007).

Chart 14. SAS output: Stepwise-regression parameter estimates.

```
                   Parameter      Standard                            Variance
Variable    DF      Estimate         Error    t Value    Pr > |t|    Inflation

wse          1   -0.00011108    0.00003177      -3.50      0.0007      1.00000
```

6. The significance level of this independent-variable coefficient is calculated under the assumption that the residuals simulate white noise. To properly make these assessments, the residuals must be tested first for stationarity and then for serial correlation. As shown in Chart 15, with p-values <.0001 for all five lags, the augmented Dickey-Fuller test results provide strong evidence that the residuals of the regression are stationary (Step H).

[28] Beal, "Information Criteria Methods."

Chart 15. SAS output: Augmented Dickey-Fuller test.

```
Type           Lags   Pr < Tau

Single Mean       0    <.0001
                  1    <.0001
                  2    <.0001
                  3    <.0001
                  4    <.0001
```

Next, with the residuals shown to be stationary, they are examined to determine if serial correlation is present.

7. The four very small p-values from the Ljung-Box test (shown in Chart 16) support rejection of the null hypothesis that there is no autocorrelation in the residuals. This provides an indication that AR and/or MA terms must be added into the model to remove the serial correlation.

Chart 16. SAS output: Autocorrelation check of residuals.

```
To     Chi-          Pr >
Lag   Square  DF    ChiSq   ------------------Autocorrelations------------------

  6    78.17   6   <.0001   -0.447   -0.055    0.336   -0.516    0.143    0.167
 12    82.04  12   <.0001   -0.093    0.038    0.005   -0.133    0.041    0.000
 18    82.41  18   <.0001    0.040   -0.009   -0.030    0.008   -0.005   -0.005
 24   106.82  24   <.0001    0.077   -0.064   -0.087    0.177   -0.237    0.241
```

As in the case of building a pure ARIMA model, the process of adding AR and/or MA terms into the regression model is driven by the significance of the ACF and PACF spikes in the residual time series (Step J). Examination of both the ACF and PACF shows that the most significant spike is in the ACF at lag 4, indicating that an MA4 term should be included in the model. Subsequent re-examination of the PACF and ACF of the times series of revised residuals indicates there are equally significant spikes at lag 1 in both the

ACF and the PACF. Consistent with the tie-breaking logic previously employed in building the pure ARIMA model, the SBCs are examined for both models. As shown in rows 2 and 3 of Chart 17, introducing an MA1 term into the model produces an SBC of −101.6, while introducing an AR1 term instead produces a more attractive SBC of −108.315. Additionally, the correlation between the MA1 and MA4 terms is 0.473, while the correlation between AR1 and MA4 is only 0.207. This explains the mitigating impact that including an MA1 term has on the t-statistic of the MA4 term, reducing it from 11.64 to 4.38. In contrast, introducing an AR1 term instead of the MA1 term only decreases the MA4 t-statistic slightly from 11.64 to 10.47.

Chart 17. ARIMAX model-building results.

Row #	Model	SBC	Term	Coefficient	t	Pr > $\|t\|$
1	MA4	−90.1125	MA4	0.82916	11.64	<.0001
2	AR1, MA4	−108.315	AR1	−0.42587	−4.99	<.0001
			MA4	0.79019	10.47	<.0001
3	MA1, MA4	−101.6	MA1	0.39005	2.39	0.0168
			MA4	0.60995	4.38	<.0001

Examination of the ACF and PACF shows there is a significant spike in the PACF at lag 2, indicating that an AR2 term should be included in the model. Further examination of the ACF and PACF indicates there are no other significant spikes.

8. After the significant (p < 0.05) AR1, AR2 and MA4 terms are entered, it is necessary to ensure that the exogenous variable(s) in the model remain significant (Step K). The

variable wse is still significant (p-value <.0001 indicated in Chart 18) and can remain in the model.

Chart 18. SAS output: ARIMAX model.

```
Parameter    Estimate       StdErr   tValue    Probt   Lag   Variable

  AR1,2    -0.2484659   0.09007034    -2.76   0.0058     2   adj_rate
  AR1,1    -0.5266464   0.09091027    -5.79   <.0001     1   adj_rate
  NUM1     -0.0000842   0.00001235    -6.82   <.0001     0   wse
  MA1,1    0.79784551   0.07491641    10.65   <.0001     4   adj_rate
```

9. Because there is only one exogenous variable present in the model, multicollinearity is not a concern (Step M). The SAS output is shown in Chart 19.

Chart 19. SAS output: Exogenous-variable multicollinearity check.

```
                           Parameter     Standard                            Variance
Variable   Label   DF       Estimate        Error    t Value    Pr > |t|    Inflation

wse        wse      1   -0.00009432   0.00002953      -3.19      0.0018      1.00000
```

10. The sign check (Step O) ensures the remaining exogenous variable has a proper sign that matches what was determined by the correlation analysis in Step F, as shown in Chart 20.

Chart 20. SAS output: Exogenous-variable sign check.

```
                                                   Pearson
                Parameter        Standard        Correlation      Sign
   Variable      Estimate          Error         Coefficient      Check
     wse       -0.00009432      0.00002953         -0.31133        Pass
```

11. Lastly, all of the p-values from the Ljung-Box test are sufficiently large (ranging from 0.1946 to 0.5857 as indicated in Chart 21), indicating that the null hypothesis of white noise residuals should not be rejected (Step Q).

Chart 21. SAS output: Autocorrelation check of residuals.

```
To     Chi-          Pr >
Lag  Square  DF    ChiSq   ------------------Autocorrelations-----------------

  6    2.34   3   0.5055    0.002    0.025    0.040   -0.015    0.002    0.126
 12   10.23   9   0.3325    0.078    0.035   -0.119   -0.190    0.017    0.048
 18   13.22  15   0.5857   -0.027   -0.095   -0.056    0.007   -0.089    0.023
 24   26.32  21   0.1946    0.006    0.042   -0.185    0.063   -0.047    0.213
```

The final ARIMAX model contains an AR1 term, an AR2, an MA4 term and one highly significant exogenous variable (*wse* unlagged), as shown Chart 22.

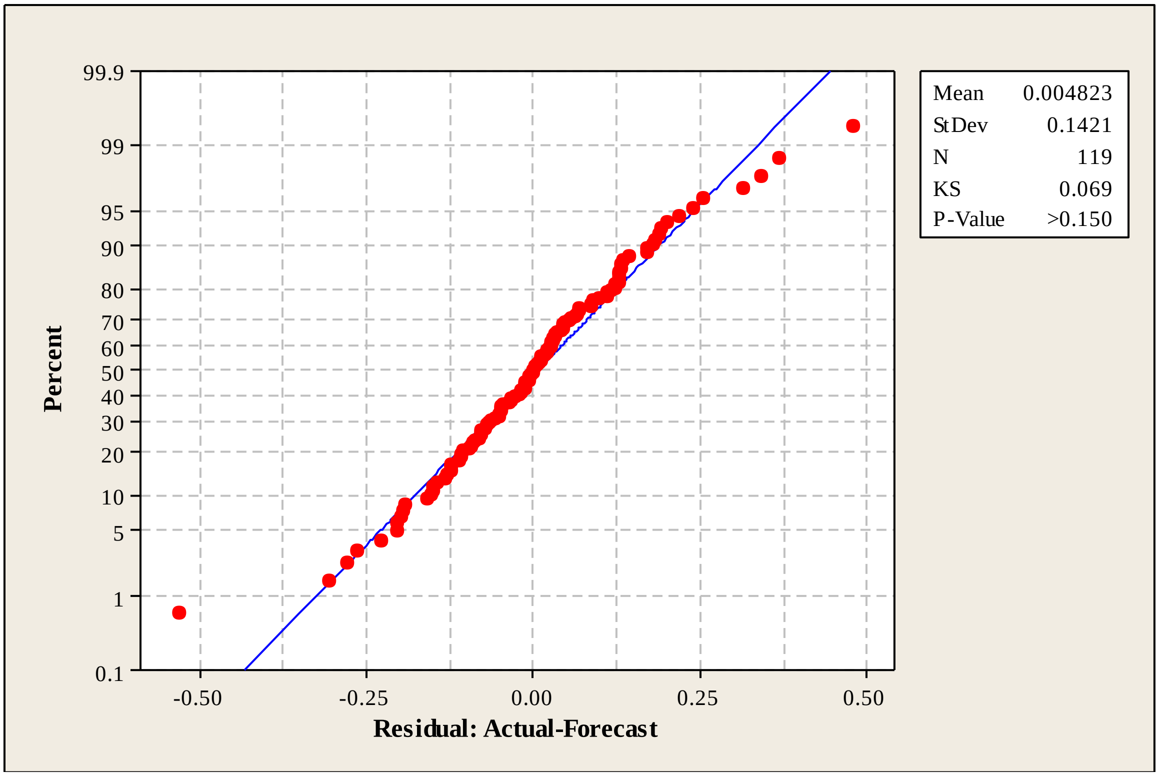
Chart 22. SAS output: Final ARIMAX model.

```
Parameter     Estimate       StdErr    tValue    Probt    Lag   Variable

  AR1,2     -0.2484659   0.09007034     -2.76   0.0058      2   adj_rate
  AR1,1     -0.5266464   0.09091027     -5.79   <.0001      1   adj_rate
  NUM1      -0.0000842   0.00001235     -6.82   <.0001      0   wse
  MA1,1     0.79784551   0.07491641     10.65   <.0001      4   adj_rate
```

Based on their t-statistics, the two most important RHS drivers in this model are clearly the MA4 and the wage-and-salary employment (wse lag 0). The MA4 makes sense because it provides an adjustment to the model based on the estimation error made in the same quarter one year earlier. The strong influence of wage and salary employment is also intuitively appealing because the availability of jobs strongly influences the level of unemployment. (Recall the WSJ article discussed earlier that referred to SSDI as “our big welfare program.”) In third place, also with a p-value of 0.0001, is the AR1 term that shows the important influence of the immediately preceding quarter. Lastly, the AR2 term with its less significant (p-value = 0.0058) coefficient captures the diluted influence of the AR1 term.

12. As in the case of the pure ARIMA model, it is necessary to ensure that the residuals of the ARIMAX model satisfy the conditions of normality and homoscedasticity. The Kolmogorov-Smirnov (K-S) test on the 119 residuals from the ARIMAX model yields a K-S statistic of 0.069 with a *p*-value of >0.150, which does not support rejection of the null hypothesis of normally distributed residuals (see Figure 9).

Figure 9. Normal probability plot of residuals.



In addition, White's test, with a *p*-value of 0.2043, indicates that the null hypothesis of homoscedasticity cannot be rejected (see Chart 23).

Chart 23. SAS output: White's test.

```
DF    Chi-Square    Pr > ChiSq

 2          3.18        0.2043
```

13. The SBC (−110.996) of the ARIMAX model (with its additional SBC-penalizing exogenous variable and additional AR2 term) is more attractive than that of the pure ARIMA model (−94.537). It is also useful to examine the goodness-of-fit measures such as the MAD and MAPE for the new ARIMAX model and to compare them to those of the pure ARIMA model to help evaluate the improved precision of the ARIMAX model.

As before with the pure ARIMA model, the four-quarter rolling MAPEs and MADs were also tabulated for the ARIMAX fit and forecast models. Unsurprisingly, the fits again outperform the holdout forecasts, although both show quite impressive goodness-of-fit results (see Chart 24).

Chart 24. ARIMAX four-quarter rolling MAPEs and MADs (Q1 1988–Q4 2012).

Estimate Type	Goodness-of-Fit Measures	n	Mean	Std. Dev.	Min.	Max.
Fit	MAPE	100	3.20	1.62	0.79	9.14
	MAD	100	0.10	0.06	0.02	0.32
Forecast	MAPE	100	4.06	2.24	0.62	12.07
	MAD	100	0.12	0.07	0.02	0.37

As seen in the performance comparison displayed in Chart 25, the ARIMAX model's average four-quarter rolling fit and forecast MAPEs show improvement at 3.20 percent and 4.06 percent, when compared to the pure ARIMA model's 3.72 percent and 5.17 percent, respectively. Further, the ARIMAX model's four-quarter rolling MADs for the fit and forecast models averaged 0.10 and 0.12, respectively, which is an attractive relative improvement over their counterparts in the pure ARIMA model (0.12 and 0.16). In fact, when comparing the four-quarter rolling MAPEs and MADs for the forecast and fit estimates, the ARIMAX model is preferable to the ARIMA model in almost every way, with its generally lower means, standard deviations, minimums and maximums (as seen by the ARIMAX/ARIMA ratios). In Chart 25, all but two of the 16 MAPE and MAD ARIMAX/ARIMA ratios for the means are less than or equal to one.

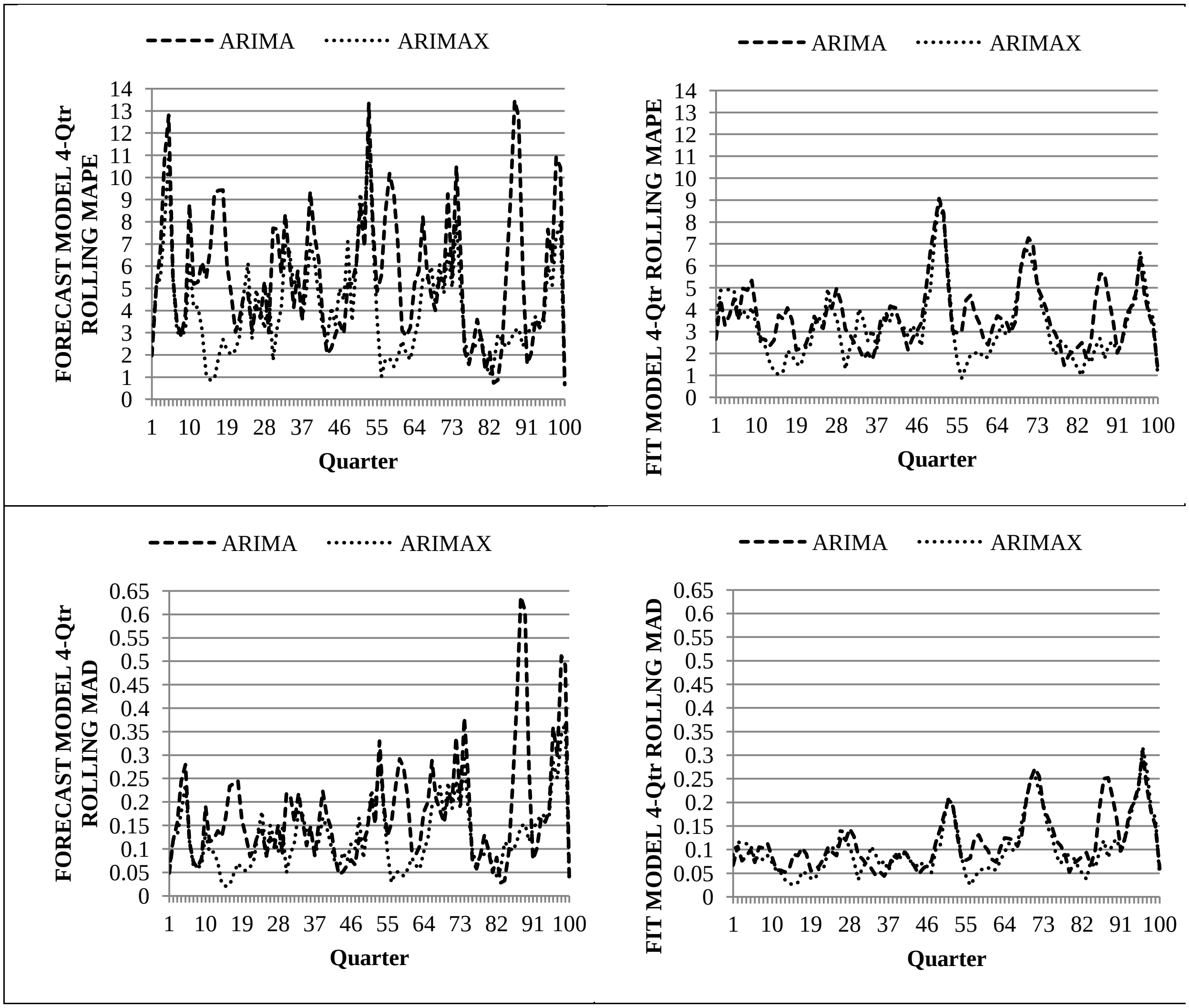
Chart 25. Performance comparison of ARIMA and ARIMAX models:

Four-quarter rolling MAPEs and MADs (100 quarters: Q1 1988–Q4 2012).

Estimate Type	Goodness-of-Fit Measures	Model	Mean	Std. Dev.	Min.	Max.
FIT	MAPE	ARIMA	3.72	1.51	1.18	9.14
		ARIMAX	3.20	1.62	0.79	9.14
		ARIMAX/ARIMA	0.86	1.07	0.67	1.00
	MAD	ARIMA	0.12	0.06	0.04	0.31
		ARIMAX	0.10	0.06	0.02	0.32
		ARIMAX/ARIMA	0.83	1.00	0.50	1.03
FORECAST	MAPE	ARIMA	5.17	2.70	1.18	13.74
		ARIMAX	4.06	2.24	0.62	12.07
		ARIMAX/ARIMA	0.79	0.83	0.53	0.88
	MAD	ARIMA	0.16	0.10	0.04	0.63
		ARIMAX	0.12	0.07	0.02	0.37
		ARIMAX/ARIMA	0.75	0.70	0.50	0.59

As a further mechanism to contrast the performance of the ARIMA and ARIMAX models, time-series graphs of the 100 four-quarter rolling MAPEs and MADs produced by the ARIMA and ARIMAX models are provided below. These graphs serve to further confirm the benefits of including exogenous variables in the model-building process.

Figure 10. Time-series graphs of four-quarter rolling MAPES and MADS: Produced by the ARIMA and ARIMAX forecast and fit models (100 quarters: Q1 1988–Q4 2012).



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