Collateralized Debt Obligations:

Reducing loss or market risk via mortgage assistance

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0: Overview/Summary

This report concerns modeling of cash flow associated within a pool of mortgages. Expanding on ideas presented in [1], this paper considers the effectiveness of exogenously infusing a limited amount of capital in a pool of mortgages. The report is comprised of the following sections:

1. Introduction: Provides a high level context for the paper and describes that a **first come first served** policy is not optimal for mortgage assistance.
2. Base Model: This section illustrates the first model to consider and is also the model from which the next two models are generalized.
3. Truncated Process: This model is the first model conditioned to default before a deterministic time.
4. A probabilistic Policy for Mortgage Assistance: Generalizes the base model by considering a policy (random) for Mortgage Assistance.
5. Observations
6. Acknowledgement
7. Appendix A: provides several of the tables referred to throughout the report.
8. Appendix B: provides several of the calculations that were useful for this work.
9. Appendix C: provides a comparison between the truncation process and the contagion setting described in [1].

1: Introduction

One of the main challenges facing mortgage relief is that there is only a finite amount of possible assistance (*i.e.* capital) and how that assistance should be provided is unclear. The main question addressed in this paper is when assistance should begin. Furthermore, for a given pool of mortgage assets, at what percentage of failures does the return on assistance become optimal, or to be more realistic, become less disastrous?

Consider the following story. After explaining to a friend the nature of this work, which can quickly become convoluted, he replied by telling me a story about his grandmother when she was 5 years old. The story is roughly as follows. In 1895, his great grandparents and family homesteaded to northern Minnesota. In that summer, the father of the family passed away and the rest of the family failed to develop enough feed for the livestock to survive the upcoming winter. The great grandmother decided to feed all the animals as best she could. As the winter progressed, she would decide if an animal was not strong enough to continue and butchered the animal. By the end of the winter, she had nearly lost their entire livestock. The family's situation was now even more severe than before the winter. My friend's grandmother would tell the story for the next 90 years, but could never understand why her mother had not decided up front how many animals could be saved with the fixed amount of feed for a typical winter. Although this story is not precisely what happens with mortgage assistance, it does capture the message that failure to identify ``cutting your losses" can lead to disaster.

We'll consider a *default* to be any event that causes the cash-flow to stop. Given one's druthers, it would be desirable to separate prepayment defaults from foreclosure defaults, but this is not typically done. The model basically provides an exogenous infusion of cash that permits (or incentizes) the asset to not default, whereby assets are modeled to have multiple defaults. The general setting gives rise to a renewal process, whereby the rate of infusion is stabilizing loss of the pool, and an optimization problem results.

We begin with a base model, and then consider modifications that *shock* the base model. For the class of assistance policies considered, it can be argued that a fixed percentage of approximately **37 percent** of the mortgage pool should default before assistance is provided. There are several important points to make regarding this statement.

1. Regarding the subprime mortgage crisis, this percentage had already been surpassed.
2. In the event that this percentage had not been exceeded, there is nothing that precludes providing assistance sooner, rather than later. The model and its policies may not incorporate all the considerations relevant to such decisions. Regardless, one should expect early assistance to return less than optimal benefits with regard to the objectives stated within this report. In cases with extremely high loan-to-values, it is very possible that such loans amount to inexpensive rent (or free) and that the obligor is not attempting to develop any equity. A relief policy needs to incorporate some degree of rejection or it runs the risk of creating a type of arbitrage. In fact, there needs to be a cost to the obligor for the assistance; otherwise, any assistance amounts to arbitrage.
3. Although not entirely related to the 37 percent threshold and the consideration that the infusion is exogenous (*i.e*. from the U.S. Government), it is of interest to also consider that it may be non-exogenous. For example, the servicer of the pool may be infusing capital into the cash flow. As the simulation indicates, the default rate can be altered in the short-term without drastically increasing the infusion rate. Depending on the transparency of infusions, this type of activity opens interesting possibilities from both the perspectives of the investor and the servicer.

1.1 First come, first served is not optimal

As pertaining to mortgage relief, some might be inclined to believe that there should be no losers. In a perfect world, this might work, but since there is only a finite amount of capital to provide assistance, one is naturally led to a constrained optimization problem. Mortgages within a pool that default sooner rather than later are more likely to have a shortened time till the next ``default."

Returning to the primary assumption that there is **not** enough capital to assist all the mortgages, it is then reasonable to ask when to begin providing assistance in order to reduce long term loses. Of course, minimizing losses from defaulting is a reasonable objective, but the timeframe in which this is achieved is also important. Regardless of whether one is concerned with a 5, 10, or 20 year time horizon, the rate at which capital is to be infused is variable over that period. This presents the possibility that early infusion of capital to assist borrowers may not achieve the overall objective over the time horizon.

The model is derived from a queuing model comprised of a collection Poisson processes and more generally, renewal processes. The goal is to address the question of the **required rate of cash infusion to ``stabilize'' losses resulting from defaults over a given time period**.

We consider letting a fixed percentage default before providing any assistance. Starting with a fixed amount of capital to provide over a period of time, if assistance is provided earlier to the asset pool, then the rate at which capital is infused is large because the renewing defaults are coming from a distribution with higher hazard rates. On the other hand, if assistance is delayed for too long, then the capital is not used sufficiently over the time period. Neither case is optimal. As a precautionary note, defining optimal is not simple; it depends on how one prioritizes the goals.

The following is a starting point for the model. Loan-to-values are established at initiation and not adjusted over time[1].

2 The Base Model

Starting with a pool of 10,000 loans with loan-to-values (LTV) ranging uniformly from 0.93 to 0.999, the asset values were all fixed to 150,000 and we assume the probability of defaulting by time *t* (see [1]) is given by

$$P(\text{time to default} \geq t) = e^{-\beta\left(\frac{l}{1-l}\right)t}, \qquad \beta = 0.004$$ [2]

The cash flow through the pool can be simulated. Although crude, it is a place to start introducing the ideas associated with this project. Summarizing,

- Asset pool size was 10,000
- Interest rates were linearly interpolated between the loan-to-values of 0.93 to 0.999 with interests ranging from 5% to 10%. For example, a loan with LTV = 0.93 was assigned an interest rate of 5% and LTV = 0.999 was assigned an interest rate of 10%.
- Loan-to-value was uniformly sampled for 3 ranges of loan-to-values. These categories are
    1. **93+:** LTV uniform on [0.93, 0.999]
    2. **96+:** LTV uniform on [0.96, 0.999]
    3. **99+:** LTV uniform on [0.99, 0.999]

[1] The model can be made more complex in directions that are more realistic, but the following starting point will hopefully suffice for introducing the approach.
[2] was calibrated from actuarial data cited in [4]

- Before infusion of any cash begins for the pool of 10,000 mortgages, a fixed percentage, *k* of the pool must default. We will be interested in the rate of cash infusion as a function of the threshold *k*.
- Assets receive no more than 6 infusions[3].
- When infusion occurs, 3 months of payments for the respective asset are accounted for allowing cash-flow to continue, and resetting the time to the next default according to the defaulting probability. In this simulation the loan-to-values were **not** recalculated based on how much principal was accrued, or change in the assessed value of the asset. The accrual of principal being paid down decreases the likelihood of default, while the decrease in the underlying asset value increases the probability of default[4].
- The maximum amount of capital that may be infused was set at 4% of the initial principle for the pool.

Initially only one realization of the process is considered, but subsequently 1000 iterations for each choice of parameter were used.

The graphs (Figure 1, 2, and 3) contain a great deal of information.

- The first graph (Figure 1) corresponds to category 93+,
- The second (Figure 2) corresponds to category 96+, and
- The third (Figure 3) corresponds to category 99+.

2.1 Charts of cash flow trajectories

A major obstacle to discussing the benefits between policies for mortgage assistance is how to compare the various time-series and different statistics related to cash flow for comparing benefits. Although tables will be useful, it will first be advantageous to have a visual chart for comparison. The task is to follow a favorable trajectory.

Remember that each chart corresponds to only one realization of the underlying stochastic process. In practice, one would have trajectories corresponding to the average infusion over time as a function of the parameters and at each point on a trajectory, a variance/error-bar. Rather than complicating the charts, the relevant parameters concerning these trajectories will be tabulated at 60 months and discussed later in this report.

Consider that there are many objectives in mortgage relief and some objectives may conflict with each other. For example

[3] Originally a nominal amount of 10 was used, and later changed to 4 groups using 3, 6, 9, and 12. The results from using no more than 6 infusions are included in the report.
[4] Adjusting loan-to-value would be more realistic, but as will be apparent the scenarios will be sufficiently complicated without this additional complexity.

- Minimize the rate at which cash is infused into a pool.
- Minimize the percentage of defaults at a given time t.
- Maximize the time after which relief is provided.
- Spend no more than a stated percentage of the principal.

The graphs show many trajectories, each indicates the amount infused for the sample asset pool as a function of the time. Focus one's attention on the trajectory (shown in black) with 63% listed next to it in Figure 1 corresponding to 93+. This means that 37% of the pool defaulted before any assistance, as described above, was provided.

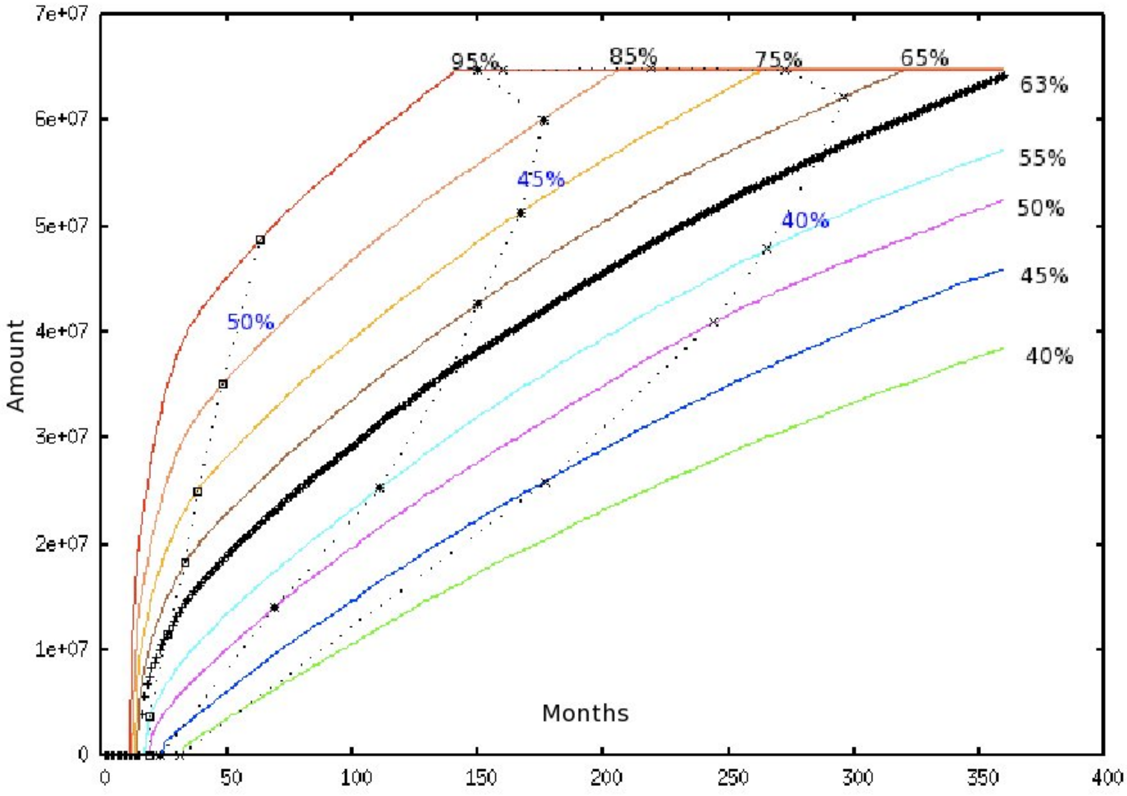

Figure 1: LTV uniformly distributed above 0.93

Transverse to these threshold trajectories there are curves indicating the percentage of the pool that is still active, or not permanently defaulted.

For scenarios of 93+, there is a drastic decrease in the rate of infusion corresponding to the 63% trajectory (37% allowed to defaulted threshold). In Table 1, the amount infused and the percentage of pool still active is compared across categories 93+, 96+, and 99+ at both 30 and 60 months.

Table 1. 30 and 60 month infusions

30 and 60 month comparison with $\beta = 0.004$				
Category 93+				
	30 months (0.38 no defaults)		60 months (0.28 no defaults)	
% threshold	infused	% active	infused	% active
95	37,761,477	55	47,875,168	51
63	13,183,941	50	21,233,487	48
40	0	41	5,030,050	39
Category 96+				
	30 months (0.18 no defaults)		60 months (0.11 no defaults)	
% threshold	infused	% active	infused	% active
95	50,543,904	33	59,407,531	27
63	27,260,151	30	35,208,242	26
40	11,890,500	27	18,030,778	25
Category 99+				
	30 months (0.01 no defaults)		60 months (0.0 no defaults)	
% threshold	infused	% active	infused	% active
95	62,714,790	11	66,034,964	02
63	45,024,183	9	50,435,911	04
40	32,472,054	8	37,016,765	04

Compare this to the trajectories of the 96+ and 99+ graphs (Figures 2 and 3). Visible in the graphs are the increased steepness of the curves and increased rates of required infusion, but for scenarios in the 99+ category 100% default is not avoided.

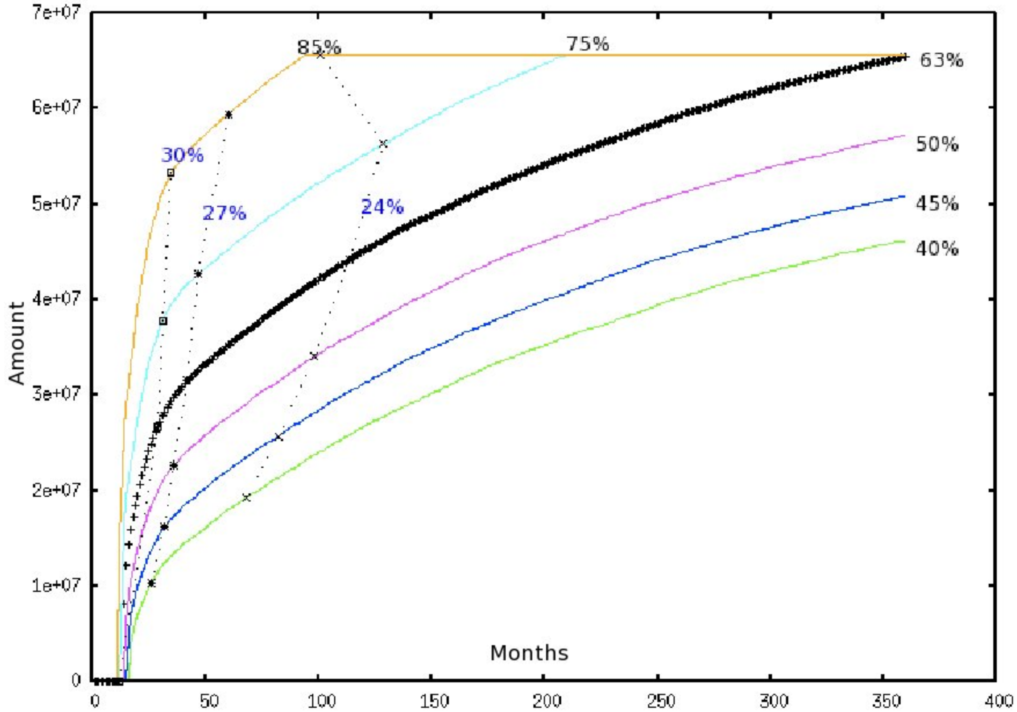

Figure 2: LTV uniformly distributed above 0.96

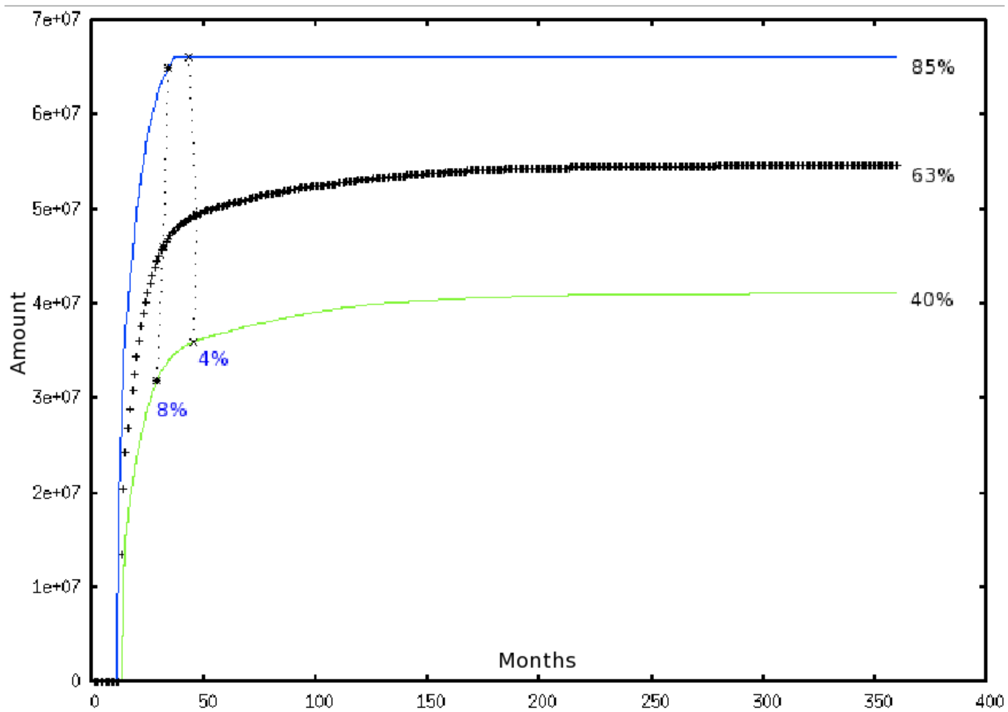

Figure 3: LTV uniformly distributed above 0.99

2.2 Some initial observations

A few observations are in order before continuing.

- The model has not been calibrated to data of recent events. In particular, the value of β may not be equal to 0.004.
- The true amount of cash infused is dependent on knowing the distribution of interest rates on loans and also the distribution of loan-to-values.
- Since it is unknown exactly how the distribution should be modeled near the singularity of LTV = 1, one might be led to consider that category 93+ or category 96+ represents the maximum risk exposure and the category 99+ is erroneous and a modeling artifact of the singularity. **Even in this case, 62 percent of the pool has defaulted within 2.5 years**.
- Consider the 93+ category in Table 1. For the 63 percent threshold - per 10,000 mortgages, if no infusion occurs, one would expect that only 38 and 28 percent of the pool to be active after 30 and 60 months, respectively, for the parameters. With infusions of 13.1 and 21.2 million, the pool maintains 50 and 48 percent at 30 and 60 months, respectively. This is $1,300 per mortgage in the first 30 months, $2,100 per mortgage in the first 60 months. **Of course, these figures are based entirely on the context of the project and that real projections need to be based on actual mortgage pools**.
- Depending on the degree of transparency of such infusions, the true default distribution could be significantly masked for a relatively small amount of infused capital over the first 60 months.

3 Truncated processes: Conditioned to default

In the next 2 sections, the initial model is extended to more complicated scenarios, but the conclusions are the same.

Recently, there has been a great deal of discussion about the modeling effects of the *tail*, or more precisely the tail behavior of a distribution. Often the context considered is when the tail represents exceedingly bad events, but in this setting the tail of the defaulting distribution represents good events that are crucial to a pool of mortgages behaving in a beneficial manner. The tail represents that the default has or will occur **after** time *t*.

Consider a sampling of a standard exponential distribution with constant rate *a*, (*i.e.* distribution function e^{-at}). One way to sample from this distribution is to first sample from a uniform distribution on the set $[0,1)$, (or U($[0,1)$). Using these uniformly distributed samples, one samples the exponential distribution by using

$$-\frac{1}{a}\ln(\,U(0,1)\,).$$

Consider shocking the base model by conditioning the defaulting distribution to default before a time *T*. This is given by the conditioned exponential distribution

$$P(t \le D \mid D < T) = \frac{\exp\{-\zeta(l)t\} - \exp\{-\zeta(l)T\}}{1 - \exp\{-\zeta(l)T\}}.$$

In the simulation involving a conditioned distribution, the defaulting distribution was conditioned to default before 20 years. The 20 year cut-off was considered because

- for a loan-to-value of 0.93, the mean time to default is approximately 19 years and
- for a loan-to-value of 0.97, removing the top 5 percent of the tail distribution amounts to removing loans defaulting after approximately 23 years.

With these considerations, the 20 year cut-off seems reasonable. In the appendix, the mean and variance for the conditioned exponential are calculated.

There are 4 main scenarios that are discussed - all using category 93+. Two of these scenarios involve the basic policy of letting a fixed percentage of loans default, one with the 20 year cut-off and one without. The other two scenarios additionally involve a probabilistic criterion that further rejects candidates from receiving assistance. In rejecting candidates after the threshold is met, there will, of course, be fewer active mortgages, but the question of whether the amount is acceptable and justifies the reduction in the infused capital begs to be asked. The policy is introduced in the next section.

Table 2 gives a summary of Tables 3, 4, 5, and 6. Tables 3 and 4 summarize the results for the simulations involving only a fixed percentage threshold. Tables 5 and 6 involve the additional probabilistic policy.

3.1 Adjustable Rate Mortgages (ARMs) and Truncated Defaulting Probability

There are two topics concerning ARMs to address. As noted earlier, the stochastic nature of loan-to-value has not been considered[5], but as it pertains to ARMs some discussion is warranted. As interest rates are reset, one can consider calculating the new payment and then determine an *effective* loan-to-value. When considering ARMs, real mortgage pools and value should be used, but it should not affect the 37% threshold. However, it will affect the timing of when the threshold is achieved. In general, this would be the ratio of two jump processes. As payments are made, the principal on the loan should be decreasing and the value can also be modeled as a jump process. More important is the non-transparency of possible *second loans* that are not modeled and pose significant default risk. It is possible to model this issue as error associated with the loan-to-value parameter. The effect would also be to shorten the time period before hitting thresholds. Since the model can be readily adjusted to provide favorable outcomes, which has been avoided in this report, the model should be considered parametrically sensitive.

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[5] Interest rates and values could be decoupled and allowed to fluctuate over time. For this report, neither is temporally stochastic and all randomness is at initiation.

Another way to model the ARM problem is by using the truncated distribution. The truncated distribution shortens the time to default and can be linked to the increase in the interest rate. We currently do not have data that could be used to complete this linking, but the 20 year truncation is fairly extreme, because it represents that 100% of the pool will default before 20 years on a 30 year note (30 year notes was also an arbitrary choice for these simulations).

One reason to consider this approach is that the non-truncated distribution forms a best case for the loan-to-value and the 37% threshold is valid for this case. In contrast, the method of calculating the effective loan-to-value would allow for resets to predict better scenarios than at origination.

4 A Probabilistic Policy for Mortgage Assistance

In addition to the fixed percentage threshold, after which assistance is possible, there is an additional criterion that now must be met. This criterion has a simple interpretation. The odds of receiving assistance are dependent on the difference between the time of default and the expected time to default. Although there are many ways to choose the dependency, we give one example that adds weight to the tail, while not excessively doing so. Furthermore, there is drastically different weighting depending on which side of the mean default time the default occurs. The introduction of a random variable associated with receiving assistance is referred to as the Probabilistic Policy, and the Base Model is referred to as the Base Policy. The Probabilistic Policy should still be considered a shock of the Base Policy.

Using the conditional probability for default, two cases arise depending on whether a default occurs before or after the mean time. Let *m* be the mean time (which depends on loan-to-value) and let t_d be the time of default.

If $m \leq t_d$, then the conditional probability that the default was greater than the mean time, given that it is less than the actual default time t_d is given by

$$P(m \leq D \,|D < t_d) = \frac{e^{-1} - e^{-\zeta(l)t_d}}{1 - e^{-\zeta(l)t_d}}.$$

Similarly if $m \geq t_d$, the conditional probability that the default was less than the mean time, given that it is greater than the actual default time t_d is given by

$$P(t_d \leq D|D < m) = \frac{e^{-\zeta(l)t_d} - e^{-1}}{1 - e^{-1}}.$$

Note that because the conditional distribution is being used and not only the joint events the normalization differs depending on the case considered. After some arithmetic, consider the utility function $U_z(\delta)$ that is derived from the previous marginal.

Let the time between defaults be denoted by δ, then at default time assistance is provided with probability $U_z(\delta)$. Note that if $\delta = m$, then $U_z(m) = z$.

$$U_z(\delta) = \begin{cases} z\left\{1 - \dfrac{e - e^{\zeta(l)\delta}}{e-1}\right\} & : \quad if\ m > \delta \\ z\left\{1 + \dfrac{1 - e\, e^{-\zeta(l)\delta}}{e - e\, e^{-\zeta(l)\delta}}\right\} & : \quad if\ m < \delta \end{cases}$$

For the simulations tabulated in this report, *z* = 0.65. In Figure 4, $U_z(\delta)$ is graphed with loan-to-value of 0.97 (*i.e.* $0.004\ \frac{0.97}{0.03} = 0.1293$).

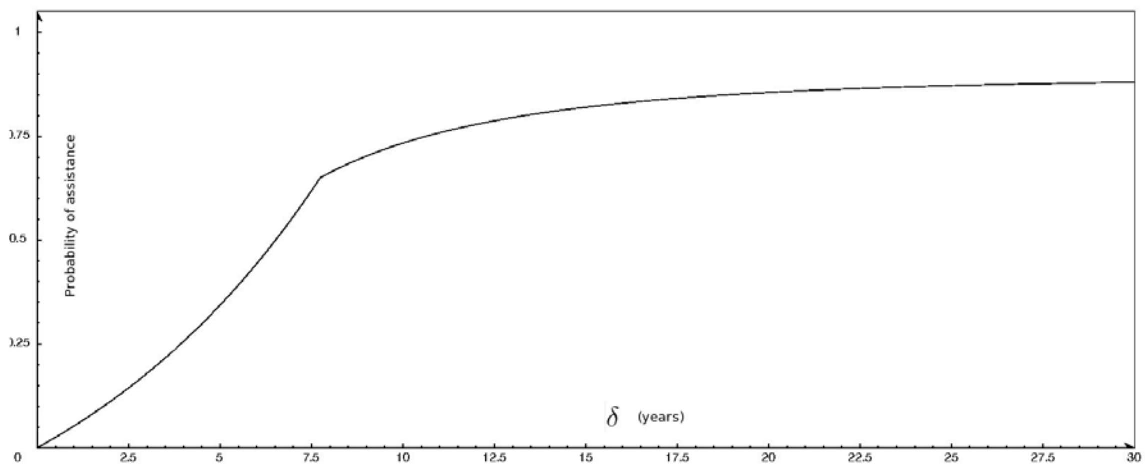

Figure 4: Probability of providing assistance

There is no a priori reason to think that this policy is optimal and depending on priorities it might not even be viewed as beneficial. For example, consider that in this example U_z was constructed to be an increasing function. It could have easily been constructed to have a maximum at $\delta = m$, whereby further reducing the probability of assistance for defaults that occur later than the mean. As time proceeds, more is paid down and there is less incentive to assist an obligor. Needless to say, this has not been a concern with the pools of mortgages that are experiencing excessive defaulting.

4.1 Pricing and triggering Issues

A very important and vast topic is that of the contractual triggers that change the cash flow on tranches. Although the original motivation for this work concerned mortgage relief, the work has touched upon the possibility that cash flow through pools of debt obligations could be altered. Knowledge of the triggers and their associated attachment points are important, and if known it is possible to customize a policy that could alter which trigger events might occur. The real question is at what cost? In cases for which the underlying mortgage pools are constructed entirely from extremely high risk loans, it is unlikely that infusion or risk masking is beneficial to anybody except possibly the borrower (see Figure 3). Even in cases of moderate high risk pools, the triggers could be placed in a manner that would make it extremely costly to mask their occurring, but it is possible and under some scenarios, it may pose a beneficial position.

Concerning pricing, one can imagine that under different circumstances that the bonds were structured to give incentive to the obligor not to default, whether by prepayment or foreclosure. Regardless of how such a contract is worded, this incentive amounts to assistance. In a scenario like this, it makes sense that the price of a tranche would reflect the potential shortfall depending on from where the capital for such an ``insurance" program comes. It is also conceivable that a pool of debt obligations could self-insure against defaults, using its own cash flow to institute a policy to avoid its own collapse. Of course, credit default swaps are a type of insurance product. Their use to insure has been more akin to Mel Brooks' story "The Producers" ([6]), than insurance. Recall that the plot of "The Producers" is that one could oversell a play/production and if the play was *guaranteed* to fail the investor would **never** have to be repaid. Although the opposite, the point is the fallacy of best laid plans.

5 Observations

If the incentive to mask risk can be argued for, then the incentive to mitigate it can also be argued. The question then becomes, at what cost an exogenous entity infuses capital? This report dealt with whether any such infusion could mitigate loss due to default within a pool of mortgages and that a fixed percentage of the pool should default before assistance to the pool is provided.

Although there have been several simplifying assumption for this work, there have also been several attributes included in the modeling that have apparently been overlooked in recent times. In all the scenarios considered, the 37 percent threshold has been robust (as also illustrated in Figures 5 and 6). Our point is not that the 37 threshold would hold for every possible scenario, but it is believed that for a broad class of default distributions this threshold would stand.

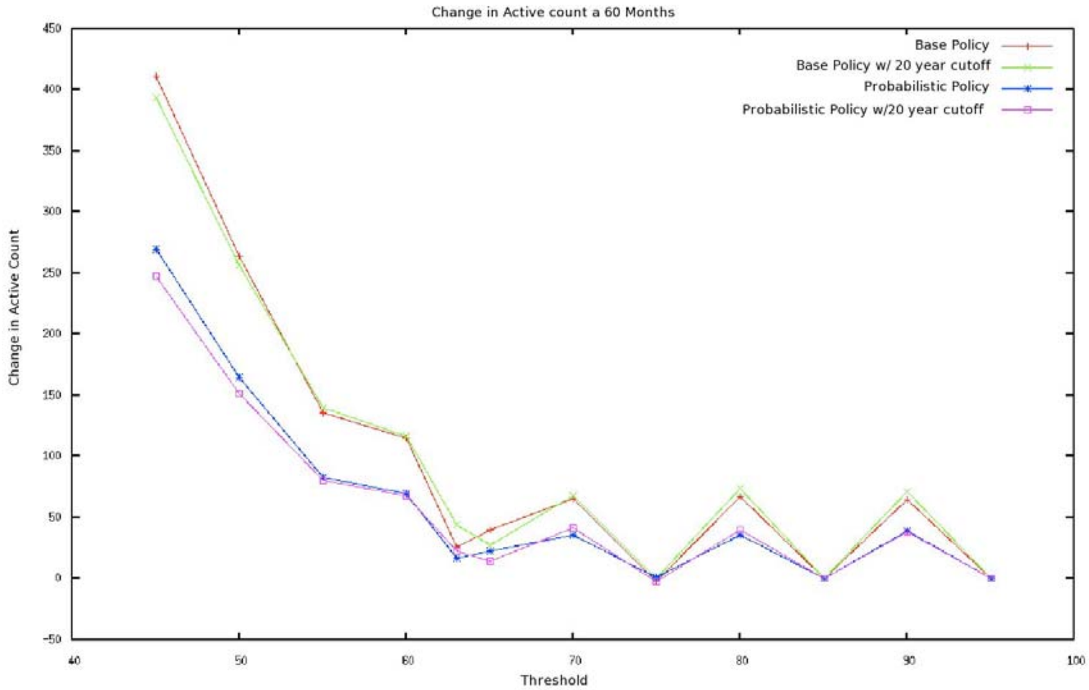

Figure 5: Change in active count

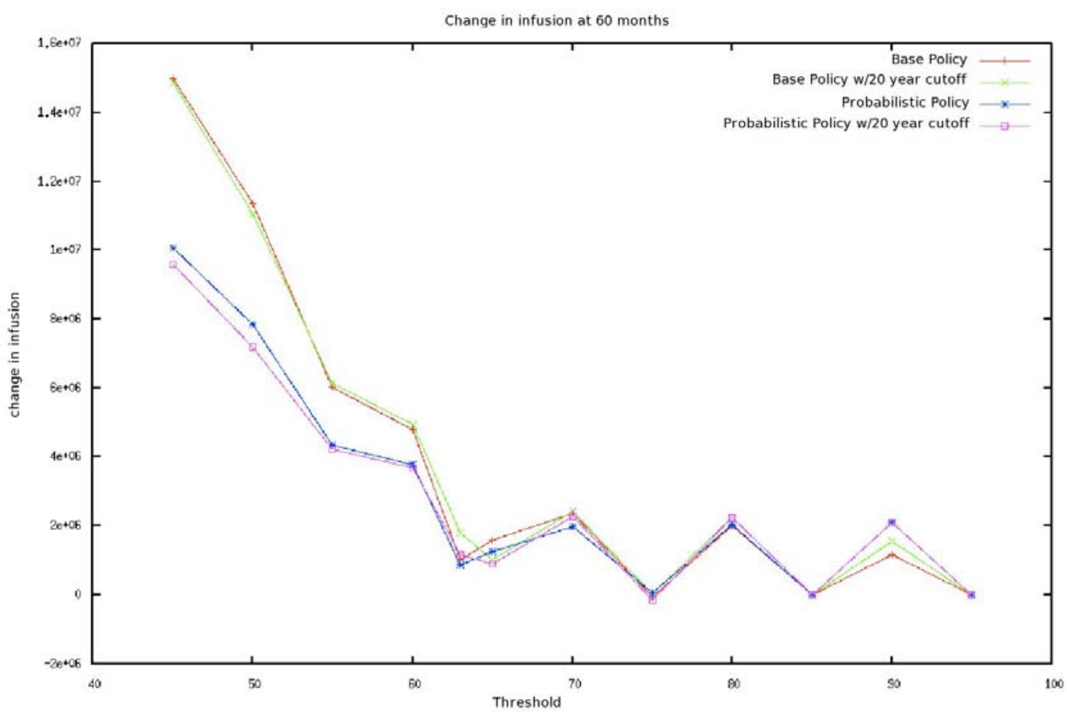

Figure 6: Change in infusion

5.1 Stochastic nature of loan-to-value

As noted previously, loan-to-value is actually stochastic over time and the work has considered it to be random, but fixed at origination of the loan. Although this work would appear to be more realistic if loan-to-value were more generally modeled, the reason to begin with it as initially fixed, is to first gain an understanding of the other dynamics. Secondly, consider the typical assumption that the credit market made: housing prices would continue to rise. Combining this with payments made, the market was actually assuming that loan-to-value would decrease over time and not increase. This being said, the initially fixed loan-to-value would have been viewed in past years as a more extreme assumption than necessary, but our assumption provides defaults for category 93+ that are commensurate with current events. That said, it's still believed that much work needs to be done to fully calibrate the parameters to real data. However, that the non-stochastic nature of loan-to-value is not a limiting factor, but rather a simplifying factor that draws attention to the fact that the law of large numbers was widely discarded by the market in regards to pools of mortgages.

In addition, this work has not directly used any correlation between assets. As noted, the effect of having a 20 year cut-off that conditioned the pool to completely default by 20 years does not change the holding threshold before which to provide assistance. It remains 63 percent active or 37 percent defaulted. Also noted, the conditioned distribution could be used instead of the contagion distribution in [1].
In this case, the fitting would result in an estimate of a time T, at which the pool has been effectively conditioned to completely default. That the parameter being fit is a time value makes it easier to interpret than an effective perturbation to a rate.

Although more appropriate as a comment to [1], there are an arbitrary large number of such perturbations that could be made, but it is the one that fits reality that is correct. Furthermore, there is nothing that precludes one function fitting better on some data and others for different data. For example, the collapsing of mortgage markets can behave differently depending on region, and there is no reason that one function needs to fit the behavior in all regions.

5.2 Conclusion

Starting with the actuarial table presented in [4], this report, together with [1] aims to convince the reader that the current mortgage crisis fit well within what should have been expected for CDOs constructed from pool of high loan-to-value ratio pools. In addition, the basic model of [3] was augmented in [1] and the defaulting distribution was further developed upon in this report. A copula methodology would be relied upon here only to fit the effective truncation of a conditioned defaulting distribution. Even in the more optimistic case of no truncation, the pools rate of defaulting can be undesirable.

Furthermore, although it may appear that the model has oversimplified certain issues, this is precisely the point; extension of this basic model to one more realistic does not reduce the risk but rather increases risk. One can take this model, and nearly any model, and make parameter choices that yield favorable results. In a certain sense, the project is actually a stronger statement for having not used any current market data. The aggregated data that was used is at least 25 years old and reported in [4].

As it pertains to current events, if mortgage assistance is to be real, the 37 percent threshold that this report theorizes has past. In the New York Times OpEd piece, [5], it was suggested that there are more cost effective means of handling the mortgage crisis. Although not the same suggested path, this work also claims that there are more cost effective paths to navigating mortgage assistance than are currently being considered. At least some degree of transparency of the assistance is needed, or the investment community is being masked from potential risks.

In practice, loans are not originated at the same time, but it is possible to calibrate the origination so that loans within a pool may be compared with one another. After calibrating the offset for origination, consider that at a given time before assistance has been introduced, the time to default can be considered as a ranking that needs to be ordered before determining when the threshold has been met.

Although the 37 percentage threshold may not be applicable to other methods of assistance, it might actually be universal. More work is needed to make that conclusion, but one should note that 0.37 is an important number (approximately e^{-1}) that arises often in probability theory. For example, e^{-1} is the percentage of applicants that should be rejected in the classic ``secretary problem''. Although not identical, there are similarities between the mortgage relief problem and the secretary problem.

The time to have been proactive has come and gone. As noted, assistance of any sort needs to have some degree of transparency otherwise risk to the investment community is being masked.

6 Acknowledgments

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7 Appendix A: Tables

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Table 2. 60 month comparisons between scenarios at 63% active

Expected number of infusions (standard dev.) Threshold 63 percent											
Policy	0 infusion	1 infusion	2 infusions	3 infusions	4 infusions	5 infusions	6 infusions	total infusion $\times 10^7$ (std $\times 10^5$)	net $\times 10^8$ (std $\times 10^6$)	active	never defaulted
Base Policy (fixed % only)	7156 (145)	1563 (51)	604 (39)	273 (27)	147 (19)	88 (14)	169 (28)	2.2 (19.0)	3.4 (2.5)	4851 (42)	2857 (47)
Base Policy (fixed % policy) w/20 year truncation	6839 (146)	1747 (52)	690 (41)	297 (27)	155 (19)	93 (15)	179 (28)	2.4 (19.0)	3.4 (2.3)	4829 (41)	2574 (45)
Probabilistic Policy only	8021 (94)	1572 (62)	299 (28)	78 (13)	23 (6)	6 (2)	1 (1)	0.97 (6.4)	3.3 (2.5)	4244 (45)	2857 (48)
Probabilistic Policy w/20 year truncation	7876 (97)	1694 (63)	316 (29)	83 (13)	25 (6)	6 (3)	1 (1)	1.03 (6.6)	3.2 (2.4)	4074 (44)	2574 (45)

Table 3. Base Policy (60 months with fixed threshold)

Base Policy Expected number of infusions (standard dev.)											
threshold	0 infusion	1 infusion	2 infusions	3 infusions	4 infusions	5 infusions	6 infusions	total infusion$\times 10^7$ (std$\times 10^5$)	net$\times 10^8$ (std$\times 10^6$)	active	never defaulted
40	8925 (46)	848 (36)	175 (17)	38 (7)	10 (3)	3 (2)	1 (1)	0.5 (2.8)	3.1 (1.5)	3900 (17)	2856 (45)
45	8461 (46)	1098 (31)	298 (20)	88 (11)	31 (6)	12 (4)	12 (4)	0.8 (4.0)	3.2 (1.7)	4310 (24)	2858 (46)
50	8046 (51)	1271 (33)	403 (21)	144 (13)	62 (9)	31 (6)	44 (8)	1.2 (5.2)	3.3 (2.1)	4574 (35)	2857 (43)
55	7711 (56)	1388 (36)	481 (23)	191 (15)	92 (11)	51 (8)	85 (11)	1.6 (6.3)	3.4 (2.5)	4710 (45)	2855 (46)
60	7285 (49)	1524 (35)	576 (24)	254 (16)	135 (12)	79 (9)	148 (13)	2.1 (5.7)	3.4 (2.7)	4825 (48)	2854 (45)
63	7156 (145)	1563 (51)	604 (39)	273 (27)	147 (19)	88 (14)	169 (28)	2.2 (19.0)	3.4 (2.5)	4851 (42)	2857 (47)
65	6964 (45)	1622 (36)	646 (24)	301 (17)	166 (13)	102 (10)	200 (14)	2.5 (5.3)	3.5 (2.9)	4891 (50)	2858 (45)
70	6544 (48)	1748 (37)	736 (27)	362 (19)	209 (14)	131 (12)	270 (17)	3.0 (6.0)	3.5 (2.8)	4957 (50)	2856 (45)
75	6542 (58)	1749 (38)	735 (27)	363 (20)	209 (15)	131 (12)	271 (17)	3.0 (7.3)	3.5 (2.8)	4956 (49)	2857 (45)
80	5986 (51)	1912 (40)	854 (28)	444 (20)	266 (17)	172 (13)	367 (19)	3.7 (6.4)	3.5 (3.0)	5023 (51)	2856 (45)
85	5986 (51)	1912 (40)	854 (28)	444 (20)	266 (17)	172 (13)	367 (19)	3.7 (6.4)	3.5 (3.0)	5023 (51)	2856 (45)
90	5241 (50)	2127 (41)	1008 (30)	553 (23)	343 (18)	226 (15)	502 (22)	4.7 (7.2)	3.5 (2.8)	5088 (47)	2854 (45)
95	5241 (50)	2127 (41)	1008 (30)	553 (23)	343 (18)	226 (15)	502 (22)	4.7 (7.2)	3.5 (2.8)	5088 (47)	2854 (45)

Table 4. Base Policy (60 months with fixed threshold) with a 20 year truncation

Base Policy w/ Truncation –Expected number of infusions (standard dev.)											
threshold	0 infusion	1 infusion	2 infusions	3 infusions	4 infusions	5 infusions	6 infusions	total infusion$\times 10^7$ (std$\times 10^5$)	net$\times 10^8$ (std$\times 10^6$)	active	never defaulted
40	8654 (44)	1036 (31)	237 (18)	53 (9)	14 (4)	4 (2)	2 (2)	0.6 (2.8)	3.0 (1.4)	3878 (18)	2574 (42)
45	8195 (44)	1269 (32)	365 (20)	105 (11)	37 (6)	15 (4)	15 (5)	0.9 (3.8)	3.2 (1.7)	4272 (26)	2573 (42)
50	7789 (45)	1436 (33)	470 (21)	159 (13)	66 (9)	33 (6)	47 (8)	1.3 (4.6)	3.3 (2.4)	4528 (42)	2574 (45)
55	7445 (78)	1555 (39)	553 (27)	209 (18)	97 (12)	53 (9)	89 (15)	1.7 (9.4)	3.3 (2.3)	4668 (39)	2572 (45)
60	7033 (47)	1688 (37)	646 (25)	268 (17)	137 (12)	79 (9)	148 (12)	2.2 (5.0)	3.4 (2.8)	4785 (48)	2571 (42)
63	6839 (146)	1747 (52)	690 (41)	297 (27)	155 (19)	93 (15)	179 (28)	2.4 (19.0)	3.4 (2.3)	4829 (41)	2574 (45)
65	6709 (50)	1786 (39)	719 (27)	315 (7)	169 (12)	102 (11)	200 (15)	2.6 (5.6)	3.4 (2.8)	4856 (50)	2574 (43)
70	6285 (51)	1914 (39)	810 (27)	377 (20)	211 (14)	132 (12)	271 (16)	3.1 (6.1)	3.4 (2.9)	4924 (49)	2573 (44)
75	6283 (59)	1915 (40)	811 (28)	378 (20)	212 (15)	132 (12)	271 (17)	3.1 (7.4)	3.4 (2.8)	4924 (50)	2573 (42)
80	5723 (48)	2078 (42)	931 (30)	461 (22)	268 (16)	172 (13)	367 (20)	3.8 (6.5)	3.5 (2.8)	4998 (50)	2574 (43)
85	5723 (48)	2078 (42)	931 (30)	461 (22)	268 (16)	172 (13)	367 (20)	3.8 (6.5)	3.5 (2.8)	4998 (50)	2574 (43)
90	4971 (51)	2298 (42)	1088 (32)	569 (23)	346 (19)	227 (15)	502 (22)	4.8 (7.1)	3.5 (2.9)	5069 (48)	2572 (43)
95	4971 (51)	2298 (42)	1088 (32)	569 (22)	346 (18)	227 (15)	502 (22)	4.8 (7.1)	3.5 (2.9)	5069 (48)	2572 (43)

Table 5. Probabilistic Policy (60 months with fixed threshold)

Probabilistic Policy –Expected number of infusions (standard dev.)											
threshold	0 infusion	1 infusion	2 infusions	3 infusions	4 infusions	5 infusions	6 infusions	total infusion×10^6 (std×10^5)	net×10^8 (std×10^6)	active	never defaulted
40	9232 (4)	725 (3)	40 (8)	2 (2)	0 (0)	0 (0)	0 (0)	2.8 (1.6)	3.0 (1.8)	3641 (24)	2857 (46)
45	8889 (4)	1001 (3)	96 (1)	11 (4)	2 (1)	0 (1)	0 (0)	4.4 (1.9)	3.1 (1.9)	3910 (32)	2855 (45)
50	8598 (41)	1208 (35)	158 (14)	28 (6)	6 (3)	1 (1)	0 (0)	6.0 (2.3)	3.2 (2.2)	4075 (40)	2856 (44)
55	8375 (43)	1353 (37)	211 (15)	45 (7)	12 (3)	3 (8)	0 (1)	7.4 (2.5)	3.2 (2.6)	4158 (47)	2855 (47)
60	8098 (42)	1526 (37)	280 (17)	70 (8)	20 (5)	5 (2)	1 (1)	9.2 (2.5)	3.3 (2.8)	4227 (51)	2855 (47)
63	8021 (94)	1572 (62)	299 (28)	78 (13)	23 (6)	6 (2)	1 (1)	9.7 (6.4)	3.3 (2.5)	4244 (45)	2857 (48)
65	7893 (43)	1650 (37)	332 (19)	90 (9)	28 (5)	7 (3)	1 (1)	10.5 (2.6)	3.3 (2.7)	4266 (49)	2857 (45)
70	7631 (45)	1808 (39)	396 (19)	116 (11)	37 (6)	10 (3)	1 (1)	12.3 (2.9)	3.3 (2.8)	4302 (49)	2857 (46)
75	7624 (53)	1811 (43)	399 (21)	117 (11)	37 (6)	10 (3)	1 (1)	12.4 (3.4)	3.3 (2.8)	4303 (49)	2856 (46)
80	7281 (46)	2014 (40)	488 (22)	152 (12)	50 (7)	13 (4)	2 (1)	14.8 (3.0)	3.3 (2.8)	4339 (50)	2855 (45)
85	7281 (46)	2014 (40)	488 (22)	152 (12)	50 (7)	13 (4)	2 (1)	14.8 (3.0)	3.3 (2.8)	4339 (50)	2855 (45)
90	6817 (48)	2288 (43)	605 (24)	201 (14)	68 (8)	19 (4)	3 (2)	18.0 (3.3)	3.3 (2.8)	4378 (48)	2855 (45)
95	6817 (48)	2288 (43)	605 (24)	201 (14)	68 (8)	19 (4)	3 (2)	18.0 (3.3)	3.3 (2.8)	4378 (48)	2855 (44)

Table 6. Probabilistic Policy (60 months with fixed threshold) with a 20 year truncation

Probabilistic Policy w/ Truncation–Expected number of infusions (standard dev.)											
threshold	0 infusion	1 infusion	2 infusions	3 infusions	4 infusions	5 infusions	6 infusions	total infusion $\times 10^6$ (std$\times 10^5$)	net $\times 10^8$ (std$\times 10^6$)	active	never defaulted
40	9093 (36)	850 (32)	53 (9)	4 (2)	0 (1)	0 (0)	0 (0)	3.4 (1.5)	2.9 (1.8)	3503 (28)	2575 (43)
45	8764 (38)	1112 (34)	108 (12)	13 (4)	2 (1)	0 (1)	0 (0)	4.9 (1.9)	3.0 (1.9)	3751 (34)	2573 (44)
50	8491 (40)	1303 (35)	168 (14)	30 (6)	7 (3)	1 (1)	0 (0)	6.4 (2.1)	3.1 (2.3)	3902 (43)	2576 (43)
55	8262 (56)	1453 (42)	222 (18)	47 (8)	12 (4)	3 (2)	0 (1)	7.8 (3.5)	3.2 (2.4)	3983 (43)	2573 (44)
60	7997 (43)	1621 (37)	285 (17)	71 (8)	21 (4)	5 (2)	1 (1)	9.5 (2.5)	3.2 (2.7)	4051 (48)	2573 (43)
63	7876 (97)	1694 (63)	316 (29)	83 (13)	25 (6)	6 (3)	1 (1)	10.3 (6.6)	3.2 (2.4)	4074 (44)	2574 (45)
65	7793 (40)	1743 (37)	338 (18)	90 (9)	27 (5)	7 (3)	1 (1)	10.9 (2.4)	3.2 (2.7)	4089 (48)	2575 (42)
70	7527 (44)	1905 (39)	404 (20)	117 (11)	37 (6)	10 (3)	1 (1)	12.7 (2.8)	3.2 (2.7)	4130 (50)	2575 (43)
75	7526 (46)	1905 (41)	405 (20)	117 (11)	36 (6)	10 (3)	1 (1)	12.7 (3.0)	3.2 (2.7)	4128 (49)	2573 (43)
80	7178 (44)	2111 (39)	493 (22)	152 (12)	50 (7)	13 (4)	2 (1)	15.0 (3.0)	3.3 (2.7)	4168 (48)	2574 (42)
85	7178 (44)	2111 (39)	493 (22)	152 (12)	50 (7)	13 (4)	2 (1)	15.0 (3.0)	3.3 (2.7)	4168 (48)	2574 (42)
90	6714 (48)	2385 (43)	611 (25)	201 (14)	68 (8)	19 (4)	3 (2)	18.3 (3.5)	3.3 (2.8)	4207 (50)	2574 (45)
95	6714 (48)	2385 (43)	611 (25)	201 (14)	68 (8)	19 (4)	3 (2)	18.3 (3.5)	3.3 (2.8)	4207 (50)	2574 (45)

8 Appendix B: Conditional defaulting distribution

This section presents the calculation for the probability of defaulting after time *t*, conditioned to also occur before a time *T*. The conditional probability distribution is needed for

- **Trimming the tail distribution**: The tail of the defaulting probability may be truncated, where shortening the time to default.
- **Ranking defaults**: The formula for the conditional probability distribution was used to construct a scoring system for defaulted loans.

Let $\zeta(l) = 0.004 \frac{l}{1-l}$, the formula for the conditional probability distribution of defaulting after time *t*, given that defaulting occurs before time *T* is given by

$$P(t \leq D \mid D < T) = \frac{e^{-\zeta(l)t} - e^{-\zeta(l)T}}{1 - e^{-\zeta(l)T}}.$$

In order to make the following calculations more readable, let the following term be singled out.

$$\epsilon = \epsilon(l, T) = \frac{e^{-\zeta(l)T}}{1 - e^{-\zeta(l)T}}$$

In cases of trimming the defaulting distribution, the conditional density is useful to compute the conditional expectation and variance. Let $p^T(dt)$ denote the probability density function, conditioned to default before time *T*.

$$p^T(dt) = \frac{\zeta(l)\, e^{-\zeta(l)t}}{1 - e^{-\zeta(l)T}} dt$$

We now summarize the calculation of the conditional expected default time

$$\boldsymbol{E}(D \mid D < T) = \int_0^T t\, p^T dt$$

$$\boldsymbol{E}(D \mid D < T) = \frac{1}{1 - e^{-\zeta(l)T}} \int_0^T t\, \zeta(l) e^{-\zeta(l)t} dt$$

Using the standard differentiation under the integral and exchanging the derivative with the integration gives

$$\frac{-\zeta(\mathrm{l})}{1 - e^{-\zeta(l)T}} \frac{\boldsymbol{d}}{\boldsymbol{d\zeta}} \int_0^T e^{-\zeta(l)t} dt = \frac{1}{\zeta(l)} - T\epsilon$$

and we have

$$\boldsymbol{E}(D \mid D < T) = \frac{1}{\zeta(l)} - T\epsilon\,.$$

Obverse that, as result of being conditioned to default before time *T*, the expected conditional mean time to default is then reduced by the term $T\epsilon$. We now give the conditional variance

$$\boldsymbol{Var}(D \mid D < T) = \int_0^T t^2 p^T(dt) \;-\; \{\boldsymbol{E}(D \mid D < T)\}^{\mathbf{2}}\,.$$

Again differentiating (twice) under the integral and exchanging the order of integration and differentiation gives

$$\int_0^T t^2 p^T(dt) \;=\; \frac{\zeta(\mathrm{l})}{1 - e^{-\zeta(l)T}} \frac{\boldsymbol{d^2}}{\boldsymbol{d\zeta^2}} \int_0^T e^{-\zeta(l)t} dt\,.$$

This term now equals

$$\frac{2}{\zeta^2(l)} - \left(T^2\epsilon \;+\; \frac{2T}{\zeta(l)}\epsilon\right).$$

Combining all the terms gives the conditional variance as

$$\boldsymbol{Var}(D \mid D < T) = \frac{1}{\zeta^2(l)} - \;T^2(\epsilon + \epsilon^2)$$

and the associated standard deviation as

$$\sigma^T(D) = \;\sqrt{\boldsymbol{Var}(D \mid D < T)}$$

$$\sigma^T(D) = \;\zeta(l)^{-1}\sqrt{1 - \zeta^2(l)T^2(\epsilon + \epsilon^2)}$$

As the truncation time *T* tends to infinity, the distribution converges to the unconditioned defaulting distribution. As the truncation time goes to 0, the conditioned distribution becomes increasingly less like the exponential distribution, but the ratio of conditioned mean time to default and the conditioned standard deviation remain bounded.

Using the freely available symbolic software package Maxima ([2]) we summarize this relationship as

$$\boldsymbol{E}(D \mid D < T)(\sqrt{3}\sigma^T(D))^{-\mathbf{1}} = \frac{1-(1+\zeta T)e^{-\zeta T}}{(1-e^{-\zeta T})\sqrt{1-(\zeta T)^2 \frac{exp(-\zeta T)}{(1-exp(-\zeta T))^2}}}$$

$$\boldsymbol{E}(D \mid D < T)(\sqrt{3}\sigma^T(D))^{-\mathbf{1}} = 1 - \frac{\zeta T}{6} + \frac{(\zeta T)^2}{40} - \frac{(\zeta T)^3}{720} + \cdots$$

From this relation, we can examine the asymptotes

$$\lim_{T\to 0} \frac{\boldsymbol{E}(D \mid D < T)}{\sqrt{3}\sigma^T(D)} = 1$$

and similarly

$$\lim_{T\to \infty} \frac{\boldsymbol{E}(D \mid D < T)}{\sigma^T(D)} = 1\,.$$

Combining these limits, there will exist an analytic function *g(T)* on *(0,T)* such that $\lim_{T\to 0} g(T) = 0$ and $\lim_{T\to\infty} g(T) = \frac{\sqrt{3}}{3} - 1$ and

$$\boldsymbol{E}(D \mid D < T) = \sqrt{3}\left(1 + g(T)\right)\sigma^T(D)\,.$$

9 Appendix C: When conditioning looks like contagion

This appendix is not really needed for this report, but since this report is a continuation of another related project with the Society of Actuaries, it is appropriate to add this section.

We start by summarizing the work reported in [1]. The main concepts behind the methods are that a contagion term can be augmented to the hazard rate of the standard copula method introduced by David Li in [3]. The augmentation is achieved by use of a perturbation term indexed by a parameter (α). The paper [1] then discusses a regression method for fitting α.

Since the writing of ``Copula Phase Transitions", one of the main questions asked has been, how does one interpret α. Alpha, α, is basically a Lyopunuv type exponent representing the rate at which contagion is spreading through the pool and is connected to how quickly a collapse of the pool of mortgages is to occur.

Between the time of writing [1] and beginning this project, consideration occurred as to whether a pool of assets could collapse at a rate increased above that attributable from the defaulting distribution, but not be due to contagion. One possible example is to consider the defaulting probability distribution conditioned to default before a time T. Simply put, conditioning can appear to act like contagion. Although a conditioned distribution is distinctly different than a contagion effect, at some level it does not matter. What matters is a way to represent a shortfall and then, if occurring, to calibrate this shortfall.

$$P(t \leq D \mid D < T) = e^{-\zeta(l)t} \frac{1 - e^{-\zeta(l)(T-t)}}{1 - e^{-\zeta(l)T}}$$

The function

$$G(t,T) = \frac{1 - e^{-\zeta(l)(T-t)}}{1 - e^{-\zeta(l)T}}$$

could be used instead of the calculated contagion term (see [1] for details)

$$CONTAG(N,\alpha,t) = \left[\frac{N - e^{\alpha t}}{N-1}\right]^{\frac{1}{\alpha}}.$$

There are several reasons to consider using *G* instead of *CONTAG*.

- G does not depend on the size of the asset pool *N*.
- T is easier to interpret than α.

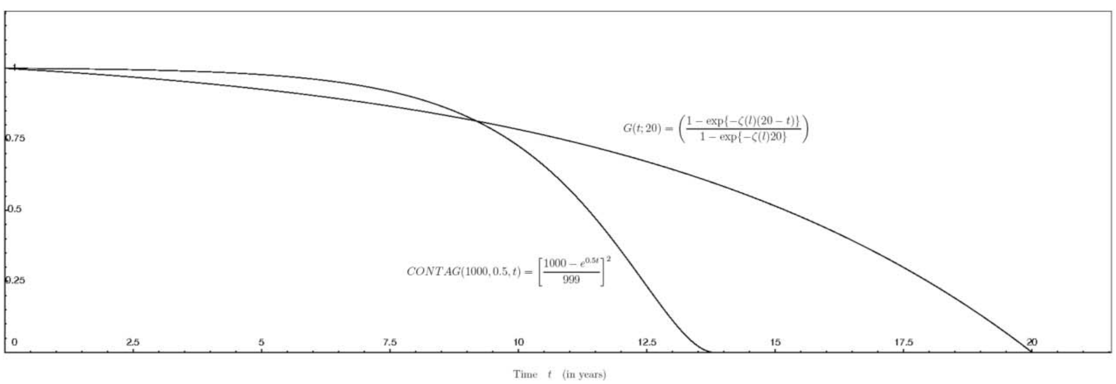

Figure 7: comparison of truncation verses contagion term

In Figure 7, the function *CONTAG*(1000, 0.5, *t*) is compared with that of *G(t;*20*)*. For these choices of parameters, the function is initially more extreme, whereas the contagion function starts with less of an effect, but then becomes more extreme rather abruptly. This is typical of phase transitions. Even if the contribution of *G(t;T)* is not to be considered a phase transition, it is not ``good effect'' and one may be led to believe that a phase transition is progress. Of course, in practice both functions (and others) could be contributed to adjusting the defaulting distribution.

9.1 Possible cause of a truncated distribution

The defaulting distribution that we have been using is certainly a simplification of the ``true" probability of defaulting. For example, it only depends on the loan-to-value. There are obvious extensions that were discussed in section 3.

Regardless of the exact nature of the defaulting probability, loan-to-value is not the only factor that one might consider. The mortgage industry has in recent times had an increased tendency to not only make loans with high loan-to-values, but also loans to people with no verifiable income where there is really no measure of debt-to-income.

A high LTV loan for an obligor with high verifiable income is less problematic than for a no documentation loan. As obligors are increasingly chosen with worse debt-to-income ranges, this amounts to conditionally sampling from an increasingly worse part of the probability distribution. Although we do not have data to support where to truncate the defaulting distribution, this has led to considering trimming/removing the tail of the distribution. Curiously, trimming the tail represents removing from the sample the best part of the sample and although the variance is reduced, it is a result of removing the part of the distribution that is most needed.

The truncation for the scenarios was chosen to be 20 years. This means that on a 30 year loan, the loan will default with probability 1 before maturity. By using a truncation of 20 years, it is interesting to explore the effects on the short term cash flow of a pool of mortgages.