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Assessing the effectiveness of local and global quadratic hedging under GARCH models

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Motivation

- Delta hedging: most popular hedging approach

$$\frac{\partial Option}{\partial S} \cdot \Delta S \approx \Delta Option$$

- Groundbreaking work due in great part to Black, Merton and Scholes
- Very powerful theoretical concept based on the paradigm of market completeness and perfect replication
- Continuous-time

Motivation

- In real life, delta hedging results in an imperfect solution
- Impossibility of trading in continuous time
- Sudden price jumps
- Market frictions
- Incomplete markets

Motivation

- Alternatives to delta hedging
- **Local risk-minimization** (Schweizer, 1988, 1991)
- **Global risk-minimization** (Schweizer, 1995)
- Perfect replication is dropped in favor of a more realistic objective: minimizing hedging costs
- Quadratic criterion
- Discrete time setting

Objective

- **Objective**: Investigate the empirical and practical relevance of local and global risk-minimization

- Three main questions:

1. Value added of global VS local quadratic hedging?
2. Choice of measure: $\mathbb{P}$ VS $\mathbb{Q}$?
3. How is hedging effectiveness impacted by model risk?

Definition of the financial market

- Discrete time: trading occurs at $\{0, 1, \ldots, T\}$
- Two traded assets: one risky stock $\{S_t\}$, and one risk-free bond $\{B_t\}$, where $B_t = \exp(rt)$
- Incomplete market
- $\mathbb{P}$: Real-world (physical, observed) probability measure
- $\mathbb{Q}$: Equivalent martingale measure (risk-neutral measure)

Choice of model for the stock asset

- Implementation: **GARCH models**
- Ubiquitous in the econometrics literature due to their strength in explaining volatility dynamics
- Have also been shown to perform well as option pricing models (Christoffersen et al., 2010)
- Easy to estimate and manipulate (in contrast to stochastic volatility jump-diffusion models)
- Allows us to relate to Badescu et al. (2014); Ortega (2012); Rémillard and Rubenthaler (2013)

GARCH model

- Risky asset $\{S_t\}_{t=0}^T$ follows GJR-GARCH(1,1):

$$\log\left(S_t/S_{t-1}\right) = \mu + \sigma_t\epsilon$$

$$\sigma_{t+1}^2 = \omega + \alpha\sigma_t^2\left(|\epsilon_t| - \gamma\epsilon_t\right)^2 + \beta\sigma_t^2$$

  where ϵ_t is a standardized Gaussian white noise under $\mathbb{P}$

- **Note**: the methodology presented in the paper can be applied to any other GARCH(1,1) specification

Contingent claim

- Our numerical analysis focuses on call options:

$$H = \max(0, S_T - K)$$

- **General problem**: risk management in an incomplete market of this derivative position with a dynamic hedging strategy and initial capital of V_0

Hedging portfolio

- Trading strategy: $\theta = \{(\theta_t^{(B)}, \theta_t^{(S)})\}_{t=0}^{T}$
- $\theta_t^{(B)}$: number of bond asset shares to be held over the time period $[t, t+1)$
- $\theta_t^{(S)}$: number of stock asset shares held over the time period $(t-1, t]$, i.e., $\theta_t^{(S)}$is determined at the previous time step
- Value of hedging portfolio:

$$V_t^{\theta} = \theta_t^{(B)} B_t + \theta_t^{(S)} S_t$$

Local quadratic hedging

- Hedging strategy that is not self-financing
- Computed through a series of local optimizations having for objective to minimize the incremental cost incurred at the next trading period
- Impose $V_T^\theta = H$. At time $t = T, T-1, \ldots,$ find the positions
$$(\theta_{t-1}^{(B)}, \theta_t^{(S)})$$
such that
$$\mathbb{E}^{\mathbb{P}}[(C_t^\theta - C_{t-1}^\theta)^2 \mid \mathcal{F}_{t-1}]$$
is minimized under $\mathbb{P}$.

Local quadratic hedging

Theorem (Local quadratic hedging)

The solution to the local quadratic hedging problem is fully determined by the following backward recursive scheme initiated at $t = T$ and $\bar{H}_T = H$:

$$\theta_t^{(S)} = \bar{\alpha}_t$$
$$\theta_{t-1}^{(B)} = B_{t-1}^{-1}\left(\bar{H}_{t-1} - \bar{\alpha}_t S_{t-1}\right)$$

where

$$\Delta_t = S_t e^{-r} - S_{t-1}$$

$$\bar{\alpha}_t = \frac{\operatorname{Cov}\left[e^{-r}\bar{H}_t, \Delta_t \mid \mathcal{F}_{t-1}\right]}{\operatorname{Var}\left[\Delta_t \mid \mathcal{F}_{t-1}\right]}$$

$$\bar{H}_{t-1} = e^{-r}\mathbb{E}\left[\bar{H}_t \mid \mathcal{F}_{t-1}\right] - \bar{\alpha}_t \mathbb{E}\left[\Delta_t \mid \mathcal{F}_{t-1}\right]$$

Hedging portfolio

- **Gains process** (discounted):

$$G_0^\theta = 0$$

$$G_t^\theta = \sum_{n=1}^{t} \theta_n^{(S)} \left(B_n^{-1} S_n - B_{n-1}^{-1} S_{n-1} \right)$$

- **Cost process** (discounted):

$$C_0^\theta = V_0^\theta$$

$$C_t^\theta = B_t^{-1} V_t^\theta - G_t^\theta$$

- Self-financing hedging strategy: constant cost

Global quadratic hedging

- Self-financing hedging strategies
- Minimize terminal squared hedging error under $\mathbb{P}$:

$$\underset{(V_0,\theta)\in\mathbb{R}\times\Theta}{\arg\min}\ \mathbb{E}^{\mathbb{P}}\left[\left(H - V_T^{\theta}\right)^2\right]$$

- Variance-optimal hedging, mean-variance hedging, global or total risk-minimization

Global quadratic hedging

Theorem (Global quadratic hedging)

The solution to the global quadratic hedging problem is fully determined by $V_0 = H_0$ and the following backward recursive scheme initiated at $t = T$, $H_T = H$ and $\nu_{T+1} = 1$:

$$\theta_t^{(S)} = \alpha_t - V_{t-1}^{\theta} b_t / a_t$$

where,

$$\begin{aligned}
\Delta_t &= S_t e^{-r} - S_{t-1} \\
a_t &= \mathbb{E}\left[\Delta_t^2 \nu_{t+1} \mid \mathcal{F}_{t-1}\right] \\
b_t &= \mathbb{E}\left[\Delta_t \nu_{t+1} \mid \mathcal{F}_{t-1}\right] \\
d_t &= e^{-r}\mathbb{E}\left[H_t \Delta_t \nu_{t+1} \mid \mathcal{F}_{t-1}\right] \\
\alpha_t &= d_t / a_t \\
\nu_t &= \mathbb{E}[(1 - \Delta_t b_t / a_t)\nu_{t+1} \mid \mathcal{F}_{t-1}] \\
H_{t-1} &= \frac{e^{-r}\mathbb{E}\left[H_t(1 - \Delta_t b_t / a_t)\nu_{t+1} \mid \mathcal{F}_{t-1}\right]}{\nu_t}
\end{aligned}$$

Global quadratic hedging

- It turns out that:

$$\text{Local under } \mathbb{P} \neq \text{Global under } \mathbb{P}$$

but

$$\text{Local under } \mathbb{Q} \Longleftrightarrow \text{Global under } \mathbb{Q}$$

- Given current state variables, the position in the stock asset obtained with the local approach is independent of the initial capital V_0, and it is also independent of previous hedging costs.
- **Not true for the global approach**.

Analysis

1. Idealized setting where there is no model risk:

$$\text{Market model} = \text{Hedging model}$$

2. Model risk experiment:

$$\text{Market model} \neq \text{Hedging model}$$

3. Empirical test

Framework

- $N = 10{,}000$ paths of the risky asset are simulated
- Terminal hedging error, $H - V_T^\theta$, is computed for each path and strategy (same V_0) assuming a daily rebalancing of the hedging portfolio
- ATM call option: $S_0 = K = 100$
- $r = 2\%$
- $T = 3$ months or 3 years
- GARCH parameters based on S&P 500 returns (1987-2010)

Experiment without model risk

- **3-month** ATM call option (initial capital = 3.44)

Model	RMSE	Semi-RMSE	95% VaR
Global $\mathbb{P}$	0.80	0.63	1.43
Local $\mathbb{P}$	1%	1%	3%
Global/local $\mathbb{Q}$	1%	1%	2%
Duan delta hedge	17%	14%	18%

Experiment without model risk

- **3-year** ATM call option (initial capital = 15.04)

Model	RMSE	Semi-RMSE	95% VaR
Global $\mathbb{P}$	1.73	1.41	2.71
Local $\mathbb{P}$	11%	8%	22%
Global/local $\mathbb{Q}$	11%	9%	23%
Duan delta hedge	41%	16%	32%

Experiment with model risk

- **3-year** ATM call option (initial capital = 15.04)
- RMSE

Model	1	2	3	4
Global $\mathbb{P}$	1.61	2.47	2.85	3.89
Local $\mathbb{P}$	16%	10%	12%	8%
Global/local $\mathbb{Q}$	16%	11%	12%	8%
Duan delta hedge	52%	10%	38%	16%

- Model 1: Regime-switching GARCH with 2 states
- Model 2: EGARCH
- Model 3: Regime-switching with 4 states,
- Model 4: Stochastic volatility with jumps

Backtest

- Option contracts issued during 2008–2010
- Rolling-window S&P 500 returns
- Results for a 3-year ATM call option

Model	RMSE	Semi-RMSE	95% VaR
Global $\mathbb{P}$	**1.49**	**1.47**	4.02
Global/local $\mathbb{Q}$	2.62	2.59	6.33
Duan delta hedge	3.11	2.83	8.32
Local Heston	2.70	2.66	7.26
B-S delta hedge	2.43	1.82	**3.47**

Key message

1. Value added of global VS local quadratic hedging?
   - Long-term maturities
   - Value added for LEAPS, market-linked CDs, VAs
2. Choice of measure: $\mathbb{P}$ VS $\mathbb{Q}$?
   - Inconsequential for local approach
   - Significant impact for global approach (choose $\mathbb{P}$)
3. How is global hedging impacted by model risk?
   - Robust to model mis-specification
   - Pareto improvement at long-term maturities