A GENERAL FORMULA FOR OPTION PRICES IN A STOCHASTIC VOLATILITY MODEL

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Paper: http://mercury.ecom.unimelb.edu.au/SITE/actwww/wps2009/No181.pdf

Followed by "Fourier Inversion Formulas in Option Pricing and Insurance"

1. THE PROBLEM

The Black-Scholes model says that the risky asset's price satisfies

$$dS_t \;=\; rS_t\,dt + \sigma S_t\,dW_t \tag{1}$$

where W is a standard Brownian motion under the risk-neutral measure.

This is often replaced with

$$dS_t \;=\; rS_t\,dt + V_t\,S_t\,dW_t, \tag{2}$$

because observed option prices do not agree with (1).

Here, $\{V_t\}$ is a stochastic process, called "stochastic volatility".

The probability distribution of S_t is usually complicated or unknown in these models. Therefore, the computation of European option prices cannot be done by a simple integration with respect to the distribution of S_t.

PROBLEM: Find an alternative way to compute European put and call prices in such models, *i.e.* to compute

$$\mathsf{E}(S_T - K)_+, \qquad \mathsf{E}(K - S_T)_+.$$

<u>N.B.</u>: "E" correspond to the risk-neutral-measure.

2. A TOOL: PARSEVAL'S THEOREM

Parseval's theorem gives conditions under which

$$\int_{-\infty}^{\infty} g(x)\,d\nu(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(-u)\hat{\nu}(u)\,du$$

where

$$\hat{g}(u) \;=\; \int_{-\infty}^{\infty} e^{iux} g(u)\,dx, \qquad \hat{\nu}(u) \;=\; \int_{-\infty}^{\infty} e^{iux}\nu(dx).$$

In option pricing, this may be applied because a European option price is:

$$\mathbf{E}g(X) \;=\; \int_{-\infty}^{\infty} g(x)\,d\mu_X(x),$$

where $\mu_X(\cdot)$ is the distribution of X (under the risk-neutral measure).

In many cases a damping factor $e^{-\alpha x}$ needs to be used, since $\hat{g}(u)$ may not be defined. Then one writes

$$g(x)\,d\mu_X(x) \;=\; e^{-\alpha x} g(x) \times e^{\alpha x}\,d\mu_X(x) \;=\; g^{(-\alpha)}(x)\,d\mu_X^{(\alpha)}(x).$$

Then

$$\mathbf{E}g(X) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g^{(-\alpha)}}(-u)\widehat{\mu_X^{(\alpha)}}(u)\,du.$$

(Ref.: Dufresne, Garrido and Morales, 2009.)

3. MAIN RESULT

The stock price satisfies

$$
\begin{aligned}
dS_t &= rS_t\,dt + V_t S_t\,dW_t \\
\iff \qquad S_t &= S_0 \exp\left(rt - \frac{U_t}{2} + \int_0^t V_s\,dW_s \right)
\end{aligned}
$$

where $$U_t = \int_0^t V_s^2\,ds.$$

The next step works if $\{V_t\}$ is independent of $\{W_t\}$ (more complicated otherwise): if we condition on V, then

$$\int_0^t V_s\,dW_s \stackrel{\mathbf{d}}{=} \sqrt{U_t}\,Z, \qquad Z \sim \mathbf{N}(0,1) \qquad (V, Z \text{ indep.})$$

Then

$$
\begin{aligned}
e^{-rT}\mathbf{E}(K-S_T)_+ &= e^{-rT}\mathbf{E}\,\mathbf{E}[(K-S_T)_+\,|\,V] \\
&= e^{-rT}\mathbf{E}[K-S_0\exp(rT-\tfrac{1}{2}U_T+\sqrt{U_T}Z)]_+ \\
&= \mathbf{E}g(U_T) \\
g(u) &= e^{-rT}\mathbf{E}[K-S_0\exp(rT+\tfrac{u}{2}+\sqrt{u}Z)]_+.
\end{aligned}
$$

The function g is the price of a European put in the Black-Scholes model.

This leads to the problem of finding the simplified expression for the Laplace transform, in the time variable, of the price of a European call or put in the Black-Scholes model.

Theorem 1. *Suppose $r \in \mathbb{R}, \sigma, K, S_0 > 0$, and let*

$$\beta = \frac{\gamma + r}{\sigma^2}, \qquad \rho = \frac{r}{\sigma^2} - \frac{1}{2}, \qquad \overline{K} = \frac{K}{S_0}$$

$$\mu_1 = \rho + \sqrt{\rho^2 + 2\beta}, \qquad \mu_2 = -\rho + \sqrt{\rho^2 + 2\beta}.$$

(a) If $\gamma > -r$, then

$$\int_0^\infty e^{-\gamma t} e^{-rt} \mathbf{E}(K - S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t})_+ \, dt$$

$$= \begin{cases} \frac{S_0}{\sigma^2\sqrt{\rho^2+2\beta}} \frac{\overline{K}^{1+\mu_1}}{\mu_1(1+\mu_1)} & \text{if } K \le S_0 \\ \frac{S_0}{\beta\sigma^2} \left[\overline{K} - \frac{2\beta}{2\beta - 2\rho - 1} + \frac{\beta \overline{K}^{1-\mu_2}}{\sqrt{\rho^2+2\beta}} \frac{1}{\mu_2(\mu_2 - 1)} \right] & \text{if } K > S_0. \end{cases}$$

(b) (Similar for the LT of a call.)

Theorem 2. *Let ν be the distribution of U_T, so that*

$$\widehat{\nu^{(\alpha)}}(u) \;=\; \mathbf{E} e^{(\alpha+iu)U_T}.$$

(a) Suppose that $\mathbf{E} e^{\alpha^* U_T} < \infty$ for some $\alpha^* > 0$. Then, for any $0 < \alpha < \alpha^*$,

$$e^{-rT}\mathbf{E}(K - S_T)_+ \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g_1^{(-\alpha)}}(-u)\widehat{\nu^{(\alpha)}}(u)\,du,$$

where, if $\overline{k} = Ke^{-rT}/S_0$,

$$\widehat{g_1^{(-\alpha)}}(-u) \;=\; \begin{cases} \dfrac{S_0\overline{k}^{(1+\sqrt{1+8\alpha+8iu})/2}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \text{if } Ke^{-rT} < S_0 \\[2ex] \dfrac{S_0(\overline{k}-1)}{\alpha+iu} + \dfrac{S_0\overline{k}^{(1-\sqrt{1+8\alpha+8iu})/2}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \text{if } Ke^{-rT} \geq S_0. \end{cases}$$

(b) (Similar integral for the price a call.)

4. NUMERICAL EXAMPLE

We applied Theorem 2 in the case where the volatility process is a Markov chain with 2 or 3 states.

In this case, option prices may be obtained by simulation as well.

Results (see paper for details):

— Theorem 2 does very well, computation times are much shorter than using simulation;

— However, some maturities give small errors, apparently due to the oscillatory integrand. More refined integration would most likely remove those errors (we use "NIntegrate" in Mathematica without any option).

FOURIER INVERSION FORMULAS

IN OPTION PRICING AND INSURANCE

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(Methodology and Computing in Applied Probability, 2009)

1. GOALS

Many authors have used Fourier inversion to compute option prices.

In particular, Lewis (2001) used Parseval's theorem to find formulas for option prices in terms of the characteristic functions of the log of the underlying. The problem here is to compute (for example) $\mathbf{E}(e^X - K)_+$ when $\mathbf{E}\, e^{iuX}$ is known.

This talk aims at widening the scope of this idea by deriving:

(1) formulas with weaker restrictions, related to classical inversion formulas for densities and distribution functions;

(2) formulas for expectations such as $\mathbf{E}(X-K)_+$ when $\mathbf{E}\, e^{iuX}$ is known (this situation occurs in option pricing and insurance).

2. SOME REFERENCES

Among many applications of Fourier inversion in option pricing:

Bakshi, G., and Madan, D.B. (2000). Spanning and derivative-security valuation. *J. Financial Economics* **55**: 205-238.

Borovkov, K., and Novikov, A. (2002). A new approach to calculating expectations for option pricing. *J. Appl. Prob.* 39: 889-895.

Carr, P., and Madan, D.B. (1999). Option valuation using the fast Fourier transform. *J. Computational Finance* **2**: 61-73.

Heston, S.L. (1993). A closed-form solution for options with stochastic volatility with application to bond and currency options. *Review of Financial Studies* **6**: 327-343.

Lewis, A. (2001). A simple option formula for general jump-diffusion and other exponential Lévy processes. OptionCity.net publications: `http://optioncity.net/pubs/ExpLevy.pdf`.

Raible, Sebastian (2000). *Lévy Processes in Finance: Theory, Numerics, and Empirical Facts*. Doctoral dissertation, Faculty of Mathematics, University of Freiburg.

3. THE PROBLEM

The no-arbitrage price of a European call option is

$$e^{-rT}\mathbf{E}(S_T - K)_+ \,,$$

where the expectation is under the risk-neutral measure.

Many models assume $X_T = \log S_T$ is not Brownian motion but a more complicated process (*e.g.* a Lévy processes).

In those cases, exact pricing of the option is often done in two steps:

(1) find the distribution of X_T, and

(2) integrate $(e^x - K)_+$ against the distribution.

It is possible to significantly shorten this, if the Fourier transform of X_T is known (which is often the case).

The expectation $\mathbf{E}(e^X - K)_+$ is an instance of

$$\mathbf{E}\, g(X) \;=\; \int_{-\infty}^{\infty} g(x)\, d\mu_X(x), \qquad (*)$$

where $\mu_X(\cdot)$ is the distribution of the variable X.

Parseval's theorem allows one to compute $(*)$ directly from the Fourier transform, without having to find the distribution of X in the first place.

4. FOURIER TRANSFORMS

Fourier transform of a function: if $h \in L^1$,

$$\hat{h}(u) := \int_{-\infty}^{\infty} h(x) e^{iux}\, dx.$$

Fourier transform of a measure:

If μ is a measure with $|\mu| < \infty$, then

$$\hat{\mu}(u) := \int_{-\infty}^{\infty} e^{iux}\, d\mu(x).$$

The characteristic function of a probability distribution μ_X is then

$$\hat{\mu}_X(u) \;=\; \mathbf{E}\, e^{iuX} \;=\; \int_{-\infty}^{\infty} e^{iux}\, d\mu_X(x).$$

5. FOURIER INVERSION

Theorem. *Suppose h is a real function which satisfies the following conditions:*

(a) $h \in L^1$ and

(b) [omitted technical conditions].

Then

$$\frac{1}{2}[h(x+) + h(x-)] \;=\; \int_{-\infty}^{\infty} e^{-iux}\hat{h}(u)\,du.$$

N.B. Last integral is a *principal value* (= Cauchy) integral.

6. PARSEVAL'S THEOREM

Theorem. *If a random variable X has distribution μ_X, then*

$$\mathbf{E}\, g(X) \;=\; \int_{-\infty}^{\infty} g(x)\,\mu_X(dx) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(-u)\hat{\mu}_X(u)\,du,$$

provided that

(i) $g \in L^1$, and

(ii) [omitted technical conditions, usually satisfied in option pricing].

7. EXPONENTIAL DAMPING (= TILTING)

Parseval's theorem cannot directly be applied to the pricing of calls and puts because the functions

$$g_1(x) = (e^x - K)_+, \qquad g_2(x) = (K - e^x)_+$$

are not in L^1 ($\Rightarrow$ Parseval's theorem does not apply).

Lewis (2001) shows how this difficulty can be avoided by modifying g_1 (or g_2). For now, assume that X has a density $f_X(\cdot)$.

For any function φ, let

$$\varphi^{(\alpha)}(x) = e^{\alpha x}\varphi(x), \qquad x \in \mathbb{R}. \qquad \text{“tilted } \varphi\text{”}$$

The Fourier transform of $\varphi^{(\alpha)}$ is denoted $\widehat{\varphi^{(\alpha)}}$.

Of course, we have: $$g(x)f_X(x) = g^{(-\alpha)}(x)f_X^{(\alpha)}(x).$$

If it happens that both $g^{(-\alpha)}$ and $f_X^{(\alpha)}$ are in L^1, then Parseval's theorem says that

$$\begin{aligned} \mathbf{E}\, g(X) &= \int_{-\infty}^{\infty} g^{(-\alpha)}(x) f_X^{(\alpha)}(x)\, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{g^{(-\alpha)}}(-u) \widehat{f_X^{(\alpha)}}(u)\, du. \end{aligned}$$

Lewis (2001) assumes that X has a density, which is not always the case in applications. We thus reformulate Lewis's result in terms of a general probability distribution μ_X:

For a measure μ and $\alpha \in \mathbb{R}$, define a new measure $\mu^{(\alpha)}$ by

$$d\mu^{(\alpha)}(x) \; = \; e^{\alpha x} \, d\mu(x).$$

The Fourier transform of $\mu^{(\alpha)}$ is denoted $\widehat{\mu^{(\alpha)}}$.

If μ_X is the distribution of X, then Parseval's theorem says that

$$\begin{aligned} \mathbf{E}\, g(X) \; &= \; \int_{-\infty}^{\infty} g^{(-\alpha)}(x) \, d\mu_X^{(\alpha)}(x) \\ &= \; \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{g^{(-\alpha)}}(-u) \widehat{\mu_X^{(\alpha)}}(u) \, du. \end{aligned}$$

Recognizing that $\widehat{\mu_X^{(\alpha)}}(u) = \mathbf{E}\, e^{(iu+\alpha)X}$, we have:

Theorem 1. *Suppose that, for a particular* $\alpha \in \mathbb{R}$,

(a) $\mathbf{E}\, e^{\alpha X} < \infty$, (Tilted distribution integrable)

(b) $g^{(-\alpha)} \in L^1$, (Tilted payoff integrable)

(c) [omitted technical condition, usually satisfied.]

Then

$$\mathbf{E}\, g(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{g^{(-\alpha)}}(-u) \widehat{\mu_X^{(\alpha)}}(u)\, du.$$

Potential problem: not always possible to find such an α.

8. LINKS WITH CLASSICAL INVERSION THEOREMS

Two classical theorems

Let X be a random variable and $F_X(x) = \mathbf{P}(X \leq x)$.

Theorem A. *If a and $a+h$ are continuity points of F_X, then*

$$F_X(a+h) - F_X(a) \;=\; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \frac{1 - e^{-iuh}}{iu} e^{-iua} \hat{\mu}_X(u)\, du.$$

Theorem B. *If F_X is continuous at $x = b$, then*

$$F_X(b) \;=\; \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{1}{iu} [e^{iub} \hat{\mu}_X(-u) - e^{-iub} \hat{\mu}_X(u)]\, du.$$

In option pricing, Theorem B leads to the well-known result:

if $\mathbf{E}\, e^X < \infty$, then

$$\mathbf{E}(e^X - K)_+ \;=\; \mathbf{E}(e^X)\Pi_1 - K\Pi_2,$$

where

$$\begin{aligned}
\Pi_1 &= \mathbf{E}\left[e^X \mathbf{1}_{\{e^X > K\}}\right] / \mathbf{E}(e^X) \\
&= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathbf{Re}\left[\frac{K^{-iu}\hat{\mu}_X(u-i)}{iu\hat{\mu}_X(-i)}\right] du \\
\Pi_2 &= \mathbf{P}\{e^X > K\}.
\end{aligned}$$

This raises the question: are there general Fourier inversion formulas for puts or calls that do not assume $\mathbf{E}\, e^{\alpha X} < \infty$ for some $\alpha \neq 0$?

- Calls: no, because $\mathbf{E}(e^X - K)_+ < \infty \quad \Longleftrightarrow \quad \mathbf{E}\, e^X < \infty$.

- Puts: yes, there is (next slide).

Theorem 2. (Fourier integral for "generalised puts")

Suppose $K, p > 0$. *For any* X, *let*

$$h(u) \;=\; \frac{\hat{\mu}_X(u)\Gamma(-iu)K^{-iu}}{\Gamma(p+1-iu)}, \quad u \in \mathbb{R}.$$

Then

$$\mathbf{E}[(K - e^X)_+]^p \;=\frac{K^p}{2} + \frac{K^p\Gamma(p+1)}{\pi}\int_0^\infty \mathbf{Re}[h(u)]\,du.$$

N.B. The gamma functions simplify in $h(\cdot)$ if $p = 1, 2, \ldots$.

9. A SLIGHTLY DIFFERENT PROBLEM

Suppose now that the payoff is not $(e^X - K)_+$ but

$$g(X) = (X - K)_+ \, ,$$

and that $\hat{\mu}_X(u) = \mathbf{E}\, e^{iuX}$ is known. (This happens with interest rate options, stop-loss insurance, ...)

For all $\alpha > 0$,

$$g^{(-\alpha)}(x) = e^{-\alpha x}(x - K)_+ \quad \Rightarrow \quad g^{(-\alpha)} \in L^1.$$

Hence, if $\mathsf{E}\, e^{\alpha X} < \infty$ for some $\alpha > 0$, then we can apply Parseval's theorem:

$$
\begin{aligned}
\mathsf{E}(X-K)_+ &= \int_{-\infty}^{\infty} g^{(-\alpha)}(x)\, d\mu_X^{(\alpha)}(x) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g^{(-\alpha)}}(-u)\widehat{\mu_X^{(\alpha)}}(u)\, du.
\end{aligned}
$$

However, in practical applications this cannot always be done, because there may not be $\alpha > 0$ such that $\mathsf{E}(e^{\alpha X}) < \infty$.

Example: $X \sim$ Compound Poisson/Pareto

Suppose X has a compound Poisson distribution

$$X = \sum_{n=1}^{N} C_n$$

where $N \sim \mathbf{Poisson}(\lambda)$ and the $\{C_n\}$ all have a Pareto distribution with density

$$f_C(x) = \frac{\beta}{(1+x)^{\beta+1}} \mathbf{1}_{\{x>0\}}.$$

Then $\mathsf{E}(e^{\alpha X}) = \infty$ for all $\alpha > 0$, though $\mathsf{E}\, X < \infty$ if $\beta > 1$.

The characteristic function of X is known:

$$\hat{\mu}_X(u) \;=\; e^{\lambda[\hat{\mu}_C(u)-1]}$$

where $\hat{\mu}_C(u)$ may be expressed in terms of the incomplete gamma function.

Exponential tilting does not work, but two alternative solutions can be found.

1st solution: Polynomial damping factors

For $c > 0$, let

$$g^{[-\beta]}(x) \;=\; (1+cx)^{-\beta} g(x), \qquad d\mu_X^{[\beta]}(x) \;=\; (1+cx)^{\beta} d\mu_X(x).$$

For the stop-loss payoff $g(x) = (x-K)_+$, and if $\beta > 1$,

$$\begin{aligned}
\widehat{g^{[-\beta]}}(u) \;&=\; \int_{\mathbb{R}} \frac{e^{iux}(x-K)_+}{(1+cx)^{\beta}}\, dx \\
&=\; \frac{e^{iuK}}{c^2(1+cK)^{\beta-2}} \Psi(2, 3-\beta, -iu(1+cK)/c).
\end{aligned}$$

(Ψ is the confluent hypergeometric function of the second kind.)

The Fourier transform of $d\mu_X^{[\beta]}(x) \;=\; (1+cx)^\beta d\mu_X(x)$ is:

$$\widehat{\mu_X^{[\beta]}}(u) \;=\; \sum_{k=0}^{\beta} \binom{n}{k} (-ci)^k \frac{\partial^k}{\partial u^k} \hat{\mu}_X(u).$$

We find:

<u>**Theorem 3**</u>. *If, for some* $\beta \in \{2, 3, \ldots\}$, $\mathbf{E}\, X^\beta < \infty$, *then*

$$\mathbf{E}(X-K)_+ \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{g^{-[\beta]}}(u)\, \widehat{\mu_X^{[\beta]}}(u)\, du.$$

2nd solution: A general formula for $\mathsf{E}(\mathbf{X}-\mathbf{K})_+$

Theorem 4. *For any X with finite mean and any K,*

$$\mathsf{E}(X-K)_+ \;=\; \frac{\mathsf{E}X}{2} + (-K)_+ + \frac{1}{\pi}\int_0^\infty \textbf{Re}\left[\frac{e^{-iuK}(1-\hat{\mu}_X(u))}{u^2}\right] du.$$

Compare with the classical formula (Theorem B):

if F_X is continuous at $x = K$, then

$$F_X(K) \;=\; \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \textbf{Re}\left[\frac{e^{-iuK}\hat{\mu}_X(u)}{iu}\right] du.$$

10. NUMERICAL ILLUSTRATION

In insurance, the "stop-loss premium" for a risk X is $\mathsf{E}(X-K)_+$. Suppose X is compound Poisson/Generalised Pareto, with

$$X = \sum_{n=1}^{N} C_n,$$

where $N \sim \mathbf{Poisson}(\lambda)$ and the claims $\{C_n\}$ have density

$$\frac{1}{\mathrm{Beta}(a,b)} \frac{x^{b-1}}{(1+x)^{a+b}} \mathbf{1}_{\{x>0\}}$$

with $a = 5$, $b = 3$. The damping factor $(1+x)^{-3}$ is applied to $g(x) = (x-K)_+$.

Stop-Loss premiums — Compound Poisson/Generalized Pareto

	$\lambda = 1$		$\lambda = 2$		$\lambda = 3$	
K	Simulated	Fourier	Simulated	Fourier	Simulated	Fourier
0	0.7493 ($\pm$0.00196)	0.75	1.4993 ($\pm$0.00277)	1.5	2.2476 ($\pm$0.00339)	2.25
0.25	0.5959 ($\pm$0.00183)	0.5962	1.2865 ($\pm$0.00270)	1.2871	2.0119 ($\pm$0.00335)	2.0140
0.5	0.4657 ($\pm$0.00169)	0.4660	1.0915 ($\pm$0.00261)	1.0914	1.7863 ($\pm$0.00330)	1.7882
1	0.2821 ($\pm$0.00141)	0.2822	0.7702 ($\pm$0.00235)	0.7702	1.3804 ($\pm$0.00313)	1.3825