On the Probability of Ruin in a Markov-modulated Risk Model

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Abstract

In this paper, we consider a Markov-modulated risk model in which the claim inter-arrivals, claim sizes and premiums influenced by an external Markovian environment process. A system of Laplace transforms of non-ruin probabilities, given the initial environment state, is established from a system of integro-differential equations. In the two-state model, explicit formulas for non-ruin probabilities are given when the initial reserve is zero or when both claim size distributions are from a rational family.

Keywords: Markov-modulated processes; Semi-Markov processes; Rational distributions; Ruin theory; Non-ruin probability

1 Introduction

The theory of ruin has been the central interest for many authors. The main objective of ruin theory is to obtain exact formulas or approximations of ruin probabilities in various kinds risk models. In this paper we are interested in the ruin probabilities in a Markov-modulated risk model. Models of this type have been investigated, e.g., by Reinhard (1984), Asmussen (1989), Rolski (1989),

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Grandell (1991), Asmussen et al. (1995) and Snoussi (2002), who studies the severity of ruin, while Bäuerle (1996) explores the expected ruin times.

Reinhard (1984) considers a class of semi-Markov risk models in which the claim frequencies and claim amounts are influenced by an external Markovian environment process. A system of integro-differential equations for the non-ruin probabilities, when the claim size distributions are exponential, is derived. In a particular case (two possible states for the environment), the solution to this system is discussed. Some other properties related to risk theory are also considered.

More recently, Reinhard and Snoussi (2001, 2002) have discussed the severity of ruin and the distribution of the surplus prior to ruin in a discrete semi-Markov risk model, respectively. Wu (1999) develops generalized bounds for the probability of ruin under a Markovian modulated risk model. Jasiulewicz (2001) considers the probability of ruin under the influence of a premium rate which varies with the level of free reserves, while Wu and Wei (2004) investigates the same problem but the premium rate varies according to the intensity of claims, in a Markovian environment.

The purpose of this paper is to obtain the explicit formulas of the probability of ruin in a Markov-modulated model where claim intensities, claim sizes and premiums vary according to a Markovian environment. The same problem is studied by Reinhard (1984), however, there are two main differences between the two papers: first, the Laplace transform approach is used to solve the system of integro-differential equations; second, the characteristic equation is fully discussed. By these, explicit formulas for the non-ruin probabilities in a two-state model are given when the initial reserve is zero or when both claim size distributions are from a rational family. Here the rational distributions include, as the special cases, Erlang, Coxian, phase-type distributions, as well as the mixture of these distributions.

2 Preliminaries

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and all the random variables defined below are on this space. Following Reinhard (1984) and Snoussi (2002), we introduce a Markov-modulated risk model involving in a Markovian environment process.

Consider a risk model in continuous time. Denote by $\{I(t);\, t \geq 0\}$ the external environment process, which influences the frequency of claims, the distribution of claims, and the rate of premiums. Suppose that $\{I(t);\, t \geq 0\}$ is a homogeneous, irreducible and recurrent Markov process with finite state space $I = \{1, 2, \ldots, m\}$. Denote by $\Lambda = (\alpha_{ij})$, with $\alpha_{ii} := -\alpha_i$, the intensity matrix of $\{I(t);\, t \geq 0\}$. The transition probability matrix of the embedded Markov chain is then given by

$$P = [p_{ij}]\,, \qquad p_{ij} = \begin{cases} 0, & i = j, \\ \frac{\alpha_{ij}}{\alpha_i}, & i \neq j, \end{cases} \qquad i,\ j \in I\,. \tag{1}$$

Further assume that at time t claims occur according to a Poisson process with constant intensity rate $\lambda_i \in \mathbb{R}^+$, when $I(t) = i$ and the corresponding claim amounts have distribution $F_i(x)$, with density function $f_i(x)$ and finite mean μ_i $(i \in I)$. Moreover, we assume that premiums are received continuously at a positive constant rate c_i during any time interval when the environment process remains in state i. Denote by W_n and X_n, respectively, the arrival time and the amount of the nth claim, and by $T_n = W_n - W_{n-1}$ the inter-arrival time of the $(n-1)$st claim and the nth claim, with $W_0 = X_0 = T_0 = 0$.

Let $J_n = I(W_n),\ n \in \mathbb{N}$, be the state of the process I at the arrival of the nth claim. Reinhard (1984) shows that when the Markov chain $\{J_n;\, n \in \mathbb{N}\}$ is irreducible and aperiodic (thus ergodic as $m < \infty$), its unique stationary probability distribution $\pi = (\pi_1, \ldots, \pi_m)$ is given by

$$\pi_i = \frac{\frac{\lambda_i \eta_i}{\alpha_i}}{\sum_{k=1}^{m} \frac{\lambda_k \eta_k}{\alpha_k}}\,, \qquad i \in I\,, \tag{2}$$

where $\eta = (\eta_1, \ldots, \eta_m)$ is the unique stationary probability distribution of the embedded Markov chain of process I, with transition probabilities given by (1).

Suppose that the sequences of random variables $\{X_n\}_{n\geq 0}$ and $\{T_n\}_{n\geq 0}$ are conditionally independent given $\{I(t);\, t \geq 0\}$.

Now define $N(t) = \sup\{n \in \mathbb{N} \,\big|\, \sum_{i=1}^{m} T_i \leq t\}$ as the number of claims that have occured before time t. The counting process $\{N(t);\, t \geq 0\}$ is called a Markov-modulated Poisson process, which is a special case of the Cox processes. It also can be seen as a Poisson process with parameters modified by the transitions of an environment process. The corresponding surplus process $\{R(t);\, t \geq 0\}$ is then

$$R(t) = u + C(t) - \sum_{n=1}^{N(t)} X_n\,, \qquad t \geq 0\,, \tag{3}$$

where $C(t)$ denotes the aggregate premium received during interval $(0,t]$ and $u\,(\geq 0)$ is the initial reserve. Let U_n be the time at which the nth transition of the environment process occurs and I_n be the state of the environment after its nth transition. Reinhard (1984) shows that

$$C(t) = \sum_{k=1}^{N_e(t)} c_{I_{k-1}}(U_k - U_{k-1}) + c_{I_{N_e(t)}}(t - T_{N_e(t)})\,, \qquad t \geq 0\,,$$

where $N_e(t) = \sup\{n \in \mathbb{N} : U_n \leq t\}$.

Define

$$T = \inf\big\{t > 0 \,\big|\, R(t) < 0\big\}\,, \qquad (\infty,\ \text{otherwise})\,,$$

to be the time of ruin and define the ultimate ruin probabilities, given that the initial environment state is i and the initial reserve is u, by

$$\Psi_i(u) = P\big\{T < \infty \,\big|\, R(0) = u,\ I(0) = i\big\}\,, \qquad i \in I\,,\ u \geq 0\,,$$

and the ultimate ruin probability in the stationary case by

$$\Psi(u) = \sum_{k=1}^{m} \pi_i \Psi_i(u)\,, \qquad u \geq 0\,.$$

Their corresponding ultimate survival probabilities, or non-ruin probabilities, are defined, for $u \geq 0$, by $\Phi_i(u) = 1 - \Psi_i(u)$, $i \in I$, and $\Phi(u) = 1 - \Psi(u)$, respectively.

Finally, we assume that the positive loading condition satisfies [see Reinhard (1984)], i.e.,

$$d = \sum_{i=1}^{m} \pi_i \left(\frac{c_i}{\lambda_i} - \mu_i\right) > 0\,, \tag{4}$$

where π_i is given by (2).

3 Laplace transforms

Reinhard (1984) derives a system of integro-differential equations about the non-ruin probabilities, $\Phi_i(u)$, for $i = 1, 2, \ldots, m$:

$$c_i \Phi_i'(u) = (\lambda_i + \alpha_i)\Phi_i(u) - \lambda_i \int_{0-}^{u} \Phi_i(u-x)\,dF_i(x) - \alpha_i \sum_{k=1}^{m} p_{ik}\Phi_k(u)\,, \qquad u \geq 0\,, \tag{5}$$

which has a unique solution such that $\Phi_i(\infty) = 1$, for $i \in I$.

Integrating (5) from 0 to t, we have

$$\begin{aligned} c_i\Phi_i(t) &= c_i\Phi_i(0) + \lambda_i \int_0^t \Phi_i(t-y)[1-F_i(y)]dy \\ &\quad + \alpha_i \int_0^t \Big[\Phi_i(u) - \sum_{k=1}^m p_{ik}\Phi_k(u)\Big]du\,, \qquad i \in I\,,\ t \geq 0\,, \end{aligned} \tag{6}$$

which is a system of Volterra integral equations, no longer the renewal type equations for $m > 1$.

Letting t goes to ∞ in (6) gives

$$\Phi_i(0) = 1 - \frac{\lambda_i\mu_i}{c_i} - \frac{\alpha_i}{c_i}\int_0^\infty \Big[\Phi_i(u) - \sum_{k=1}^m p_{ik}\Phi_k(u)\Big]du\,, \qquad i \in I\,, \tag{7}$$

which does not give an explicit value for the probabilities $\Phi_i(0)$ as in the classical case ($m = 1$).

We now apply Laplace transforms to solve the system of equations (6). Let $\hat{\Phi}_i$ and $\hat{f}_i$ be the Laplace transforms of Φ_i and f_i, respectively, i.e.,

$$\hat{\Phi}_i(s) = \int_0^\infty e^{-su}\Phi_i(u)du\,, \qquad \hat{f}_i(s) = \int_0^\infty e^{-su}f_i(u)du\,, \qquad i \in I\,.$$

Taking Laplace transforms on both sides of equation (6) yields

$$\left[s - \frac{\lambda_i + \alpha_i}{c_i} + \frac{\lambda_i}{c_i}\hat{f}_i(s)\right]\hat{\Phi}_i(s) + \frac{\alpha_i}{c_i}\sum_{k=1}^m p_{ik}\hat{\Phi}_k(s) = \Phi_i(0)\,, \qquad i \in I$$

or in a matrix form

$$A(s)\hat{\mathbf{\Phi}}(s) = \mathbf{\Phi}(0)\,, \tag{8}$$

where

$$A(s) = \begin{bmatrix} s - \frac{\lambda_1(1-\hat{f}_1(s))+\alpha_1}{c_1} & & \\ & \ddots & \\ & & s - \frac{\lambda_m(1-\hat{f}_m(s))+\alpha_m}{c_m} \end{bmatrix} + \begin{bmatrix} \frac{\alpha_i}{c_i} & & \\ & \ddots & \\ & & \frac{\alpha_m}{c_m} \end{bmatrix} P\,, \tag{9}$$

$\hat{\mathbf{\Phi}}(s) = \left[\hat{\Phi}_1(s), \ldots, \hat{\Phi}_m(s)\right]^T$, $\mathbf{\Phi}(0) = [\Phi_1(0), \ldots, \Phi_m(0)]^T$, and P is given by (1), with $p_{ii} = 0$, for $i \in I$.

Then $\hat{\mathbf{\Phi}}(s)$ can be solved as $\hat{\mathbf{\Phi}}(s) = [A(s)]^{-1}\mathbf{\Phi}(0)$, and

$$det[A(s)] = 0\,, \tag{10}$$

is the characteristic equation of (8).

4 The results on a two-state model

In this section, we derive explicit expressions for non-ruin probabilities. By discussing analytically the roots of equation (10), the Laplace transform of $\hat{\mathbf{\Phi}}(\mathbf{s})$ can be inverted for certain types of claim size distributions.

Now we consider the case when $m = 2$, that is $\{I(t);\, t \geq 0\}$ is a two-state Markov process, which reflects the random environmental effects due to "normal" vs. "abnormal", or "high season" vs. "low season" conditions. The unique stationary probability distribution π_i can be obtained from (2) as

$$\pi_i = \frac{\frac{\lambda_i}{\alpha_i}}{\frac{\lambda_1}{\alpha_1} + \frac{\lambda_2}{\alpha_2}}\,, \qquad i = 1, 2\,,$$

and the positive loading condition (4) becomes

$$d = \frac{\frac{\lambda_1}{\alpha_1}\left(\frac{c_1}{\lambda_1} - \mu_1\right) + \frac{\lambda_2}{\alpha_2}\left(\frac{c_2}{\lambda_2} - \mu_2\right)}{\frac{\lambda_1}{\alpha_1} + \frac{\lambda_2}{\alpha_2}} > 0\,. \tag{11}$$

In this case matrix (9) has the form

$$A(s) = \begin{bmatrix} s - \frac{\lambda_1+\alpha_1}{c_1} + \frac{\lambda_1}{c_1}\hat{f}_1(s) & \frac{\alpha_1}{c_1} \\ \frac{\alpha_2}{c_2} & s - \frac{\lambda_2+\alpha_2}{c_2} + \frac{\lambda_2}{c_2}\hat{f}_2(s) \end{bmatrix},$$

and the characteristic equation (10) is of the form

$$Q(s) := \left[s - \frac{\lambda_1 + \alpha_1}{c_1} + \frac{\lambda_1}{c_1}\hat{f}_1(s)\right]\left[s - \frac{\lambda_2 + \alpha_2}{c_2} + \frac{\lambda_2}{c_2}\hat{f}_2(s)\right] = \frac{\alpha_1\alpha_2}{c_1c_2}\,. \tag{12}$$

Note that $s = 0$ is one root of equation (12). Following theorem shows that it also has one and only one positive root, which plays the key role in deriving the non-ruin probabilities $\Phi_i(u)$.

Theorem 1 Characteristic equation (12) has exactly one positive real root, say ρ, on the right half complex plane.

Now equation (8) has the form

$$\begin{bmatrix} s-\frac{\lambda_1+\alpha_1}{c_1}+\frac{\lambda_1}{c_1}\hat{f}_1(s) & \frac{\alpha_1}{c_1} \\ \frac{\alpha_2}{c_2} & s-\frac{\lambda_2+\alpha_2}{c_2}+\frac{\lambda_2}{c_2}\hat{f}_2(s) \end{bmatrix} \begin{bmatrix} \hat{\Phi}_1(s) \\ \hat{\Phi}_2(s) \end{bmatrix} = \begin{bmatrix} \Phi_1(0) \\ \Phi_2(0) \end{bmatrix},$$

or

$$\begin{cases} \hat{\Phi}_1(s) = \frac{\Phi_1(0)\left[s-\frac{\lambda_2+\alpha_2}{c_2}+\frac{\lambda_2}{c_2}\hat{f}_2(s)\right]-\Phi_2(0)\frac{\alpha_1}{c_1}}{Q(s)-\frac{\alpha_1\alpha_2}{c_1c_2}} \\ \hat{\Phi}_2(s) = \frac{\Phi_2(0)\left[s-\frac{\lambda_1+\alpha_1}{c_1}+\frac{\lambda_1}{c_1}\hat{f}_1(s)\right]-\Phi_1(0)\frac{\alpha_2}{c_2}}{Q(s)-\frac{\alpha_1\alpha_2}{c_1c_2}} \end{cases}. \tag{13}$$

Since $\hat{\Phi}_1(s)$ and $\hat{\Phi}_2(s)$ are finite for all s with $\Re(s) \geq 0$ and $Q(\rho) = \frac{\alpha_1\alpha_2}{c_1c_2}$, we have that both the numerators in (13) are zero when $s = \rho$, i.e.,

$$\Phi_1(0)\left[\rho-\frac{\lambda_2+\alpha_2}{c_2}+\frac{\lambda_2}{c_2}\hat{f}_2(\rho)\right] = \Phi_2(0)\frac{\alpha_1}{c_1}. \tag{14}$$

Then (13) can be rewritten as

$$\begin{cases} \hat{\Phi}_1(s) = \frac{\Phi_1(0)\left[\left(s-\rho\right)+\frac{\lambda_2}{c_2}\left(\hat{f}_2(s)-\hat{f}_2(\rho)\right)\right]}{Q(s)-\frac{\alpha_1\alpha_2}{c_1c_2}}, \\ \hat{\Phi}_2(s) = \frac{\Phi_2(0)\left[\left(s-\rho\right)+\frac{\lambda_1}{c_1}\left(\hat{f}_1(s)-\hat{f}_1(\rho)\right)\right]}{Q(s)-\frac{\alpha_1\alpha_2}{c_1c_2}}. \end{cases} \tag{15}$$

On the other hand, equation (7) gives

$$\frac{\alpha_2}{c_2}\Phi_1(0)+\frac{\alpha_1}{c_1}\Phi_2(0) = \frac{\alpha_1}{c_1}\left(1-\frac{\lambda_2\mu_2}{c_2}\right)+\frac{\alpha_2}{c_2}\left(1-\frac{\lambda_1\mu_1}{c_1}\right). \tag{16}$$

Combining (14) with (16), we get

Theorem 2 For risk model given by (3), with $m = 2$ and $d > 0$, the non-ruin probabilities when the initial reserve is zero are given by

$$\begin{cases} \Phi_1(0) = \frac{\frac{\alpha_1}{c_1}\left(1-\frac{\lambda_2\mu_2}{c_2}\right)+\frac{\alpha_2}{c_2}\left(1-\frac{\lambda_1\mu_1}{c_1}\right)}{\rho-\frac{\lambda_2}{c_2}\left[1-\hat{f}_2(\rho)\right]}, \\ \Phi_2(0) = \frac{\frac{\alpha_1}{c_1}\left(1-\frac{\lambda_2\mu_2}{c_2}\right)+\frac{\alpha_2}{c_2}\left(1-\frac{\lambda_1\mu_1}{c_1}\right)}{\rho-\frac{\lambda_1}{c_1}\left[1-\hat{f}_1(\rho)\right]}. \end{cases} \tag{17}$$

We now consider the case where the claim size distributions f_1 and f_2 are from a rational family, namely, their Laplace transformations are rational functions:

$$\hat{f}_1(s) = \frac{p_{k-1}(s)}{p_k(s)}\,, \qquad \hat{f}_2(s) = \frac{q_{l-1}(s)}{q_l(s)}\,, \qquad k,\ l \in \mathbb{N}^+\,, \tag{18}$$

where $p_{k-1}(s)$ and $q_{l-1}(s)$ are polynomials of degrees $k-1$ and $l-1$ or less, respectively, while $p_k(s)$, $q_l(s)$ are polynomials of degrees k and l, with only negative roots, satisfying $p_{k-1}(0) = p_k(0)$ and $q_{l-1}(0) = q_l(0)$. This general class of distributions includes, as special cases, the Erlang, Coxian and phase-type distributions, as well as mixtures of these [see Cohen (1982) and Tijms (1994)].

It turns out that equations in (15) can be transformed to rational expressions by multiplying both numerators and denominators by $p_k(s)q_l(s)$:

$$\hat{\Phi}_1(s) = \frac{\Phi_1(0)(s-\rho)p_k(s)\left\{q_l(s) + \frac{\lambda_2}{c_2}\left(q_{l-1}[s,\rho] - \frac{q_{l-1}(\rho)}{q_l(\rho)}q_l[s,\rho]\right)\right\}}{p_k(s)q_l(s)[Q(s) - \frac{\alpha_1\alpha_2}{c_1c_2}]}\,, \tag{19}$$

$$\hat{\Phi}_2(s) = \frac{\Phi_2(0)(s-\rho)q_l(s)\left\{p_k(s) + \frac{\lambda_1}{c_1}\left(p_{k-1}[s,\rho] - \frac{p_{k-1}(\rho)}{p_k(\rho)}p_k[s,\rho]\right)\right\}}{p_k(s)q_l(s)[Q(s) - \frac{\alpha_1\alpha_2}{c_1c_2}]}\,, \tag{20}$$

where $p_{k-1}[s,\rho] := \frac{p_{k-1}(s)-p_{k-1}(\rho)}{s-\rho}$, a polynomial of degree $k-2$, is the first order divided difference of $p_{k-1}(s)$ with respect to ρ, and $p_k[s,\rho]$, $q_{l-1}[s,\rho]$ and $q_l[s,\rho]$ have the similar definitions. It is clear that both numerators of (19) and (20) are now polynomials of degree $k+l+1$.

For simplicity, let $D_{k+l+2}(s)$ be the common denominator of (19) and (20), which is clearly a polynomial of degree $k+l+2$ with the leading coefficient 1. Then equation $D_{k+l+2}(s) = 0$, i.e.,

$$\begin{aligned}&\left[\left(s - \frac{\lambda_1+\alpha_1}{c_1}\right)p_k(s) + \frac{\lambda_1}{c_1}p_{k-1}(s)\right]\left[\left(s - \frac{\lambda_2+\alpha_2}{c_2}\right)q_l(s) + \frac{\lambda_2}{c_2}q_{l-1}(s)\right] \\ &- \frac{\alpha_1\alpha_2}{c_1c_2}p_k(s)q_l(s) = 0\end{aligned} \tag{21}$$

has $k+l+2$ roots on the complex plane and all of them are in pairs of conjugate forms. Note that $s=0$ and $s=\rho$ are of two roots, then

$$D_{k+l+2}(s) = s\,(s-\rho)\prod_{i=1}^{k+l}(s+R_i)\,.$$

We remark that all R_i's have a positive real parts, since, otherwise, it is also the root of the characteristic equation, which is a contradiction to the conclusion that there is only one root to it.

Then (19) and (20) can be simplified to

$$\begin{cases} \hat{\Phi}_1(s) = \frac{\Phi_1(0)p_k(s)\left\{q_l(s)+\frac{\lambda_2}{c_2}\left(q_{l-1}[s,\rho]-\frac{q_{l-1}(\rho)}{q_l(\rho)}q_l[s,\rho]\right)\right\}}{s\prod_{i=1}^{k+l}(s+R_i)} = \frac{\Phi_1(0)g_{k+l}(s)}{s\prod_{i=1}^{k+l}(s+R_i)}\,, \\ \hat{\Phi}_2(s) = \frac{\Phi_2(0)q_l(s)\left\{p_k(s)+\frac{\lambda_1}{c_1}\left(p_{k-1}[s,\rho]-\frac{p_{k-1}(\rho)}{p_k(\rho)}p_k[s,\rho]\right)\right\}}{s\prod_{i=1}^{k+l}(s+R_i)} = \frac{\Phi_2(0)h_{k+l}(s)}{s\prod_{i=1}^{k+l}(s+R_i)}\,, \end{cases}$$

where

$$\begin{aligned} g_{k+l}(s) &= p_k(s)\left\{q_l(s)+\frac{\lambda_2}{c_2}\left(q_{l-1}[s,\rho]-\frac{q_{l-1}(\rho)}{q_l(\rho)}q_l[s,\rho]\right)\right\}, \\ h_{k+l}(s) &= q_l(s)\left\{p_k(s)+\frac{\lambda_1}{c_1}\left(p_{k-1}[s,\rho]-\frac{p_{k-1}(\rho)}{p_k(\rho)}p_k[s,\rho]\right)\right\}. \end{aligned}$$

Then if R_i, $i = 1, 2, \ldots, k+l$, are distinct numbers, we obtain the following theorem.

Theorem 3 For risk models given by (3), with $m = 2$ and $d > 0$, if the claim size distributions are of rational family (18), the non-ruin probabilities are given by

$$\Phi_1(u) = 1 + \Phi_1(0)\sum_{i=1}^{k+l} g_i e^{-R_i u}\,, \qquad \Phi_2(u) = 1 + \Phi_2(0)\sum_{i=1}^{k+l} h_i e^{-R_i u}\,, \tag{22}$$

where $-R_1, -R_2, \ldots, -R_{k+l}$, are distinct roots of equation (21), with negative real parts, and $\Phi_1(0)$ and $\Phi_2(0)$ are given by (17), while g_i, h_i are of the forms

$$g_i = \frac{-g_{k+l}(-R_i)}{R_i\prod_{j=1,j\neq i}^{k+l}(R_j-R_i)}\,,\ h_i = \frac{-h_{k+l}(-R_i)}{R_i\prod_{j=1,j\neq i}^{k+l}(R_j-R_i)}\,,\ \ i = 1, 2, \ldots, k+l\,. \tag{23}$$

We remark that if some of R_i's come in pairs of complex forms, the non-ruin probabilities may contain damped trigonometric functions.

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