Quantitative Finance and Investments Core Formula Sheet

Spring 2014/Fall 2014

(some typos fixed for the Fall 2014 version)

Morning and afternoon exam booklets will include a formula package identical to the one attached to this study note. The exam committee believe that by providing many key formulas, candidates will be able to focus more of their exam preparation time on the application of the formulas and concepts to demonstrate their understanding of the syllabus material and less time on the memorization of the formulas.

The formula sheet was developed sequentially by reviewing the syllabus material for each major syllabus topic. Candidates should be able to follow the flow of the formula package easily. We recommend that candidates use the formula package concurrently with the syllabus material. Not every formula in the syllabus is in the formula package. **Candidates are responsible for all formulas on the syllabus, including those not on the formula sheet.**

Candidates should carefully observe the sometimes subtle differences in formulas and their application to slightly different situations. For example, there are several versions of the Black-Scholes-Merton option pricing formula to differentiate between instruments paying dividends, tied to an index, etc. Candidates will be expected to recognize the correct formula to apply in a specific situation of an exam question.

Candidates will note that the formula package does not generally provide names or definitions of the formula or symbols used in the formula. With the wide variety of references and authors of the syllabus, candidates should recognize that the letter conventions and use of symbols may vary from one part of the syllabus to another and thus from one formula to another.

We trust that you will find the inclusion of the formula package to be a valuable study aide that will allow for more of your preparation time to be spent on mastering the learning objectives and learning outcomes.

In sources where some equations are numbered and others are not (nn) denotes that there is no number assigned to that particular equation.

An Introduction to the Mathematics of Financial Derivatives, 2nd Edition, S. Neftci

Chapter 2

(10) $$\begin{bmatrix} 1 \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} (1+r) & (1+r) \\ S_1(t+1) & S_2(t+1) \\ C_1(t+1) & C_2(t+1) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

(66) $$S_t = \frac{1}{1+r}\left[\tilde{P}_{up}(S_t + \sigma\sqrt{\Delta}) + \tilde{P}_{down}(S_t - \sigma\sqrt{\Delta})\right]$$

(67) $$C_t = \frac{1}{(1+r)}\left[\tilde{P}_{up}C^{up}_{t+\Delta} + \tilde{P}_{down}C^{down}_{t+\Delta}\right]$$

(70) $$S = \frac{(1+d)}{(1+r)}\left[S^u\tilde{P}^u + S^d\tilde{P}^d\right]$$

(71) $$C = \frac{1}{(1+r)}\left[C^u\tilde{P}^u + C^d\tilde{P}^d\right]$$

(nn) $$E^{\tilde{P}}\left[\frac{C_{t+\Delta}}{C_t}\right] \cong 1 + r\Delta$$

Chapter 3

(37) $$\sum_{i=1}^{n} f\left(\frac{t_i + t_{i-1}}{2}\right)(t_i - t_{i-1}) \to \int_0^T f(s)ds$$

(49) $$\int_0^T g(s)df(s) \cong \sum_{i=1}^{n} g\left(\frac{t_i + t_{i-1}}{2}\right)(f(t_i) - f(t_{i-1}))$$

Chapter 4

(24) $$dF(t) = F_s dS_t + F_r dr_t + F_t dt + \frac{1}{2}F_{ss}dS_t^2 + \frac{1}{2}F_{rr}dr_t^2 + F_{sr}dS_t dr_t$$

Chapter 5

(11) $$E[S_t|I_u] = \int_{-\infty}^{\infty} S_t f(S_t|I_u)dS_t, \qquad u < t$$

(18) $$P(\Delta F(t) = +a\sqrt{\Delta}) = p$$

(19) $$P(\Delta F(t) = -a\sqrt{\Delta}) = (1-p)$$

(40) $$P(X_{t+s} \le x_{t+s}|x_t, \ldots, x_1) = P(X_{t+s} \le x_{t+s}|x_t)$$

(41) $$r_{t+\Delta} - r_t = E[(r_{t+\Delta} - r_t)|I_t] + \sigma(I_t, t)\Delta W_t$$

(44) $$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dW_t$$

(45) $$\begin{bmatrix} r_{t+\Delta} \\ R_{t+\Delta} \end{bmatrix} = \begin{bmatrix} \alpha_1 r_t + \beta_1 R_t \\ \alpha_2 r_t + \beta_2 R_t \end{bmatrix} + \begin{bmatrix} \sigma_1 W^1_{t+\Delta} \\ \sigma_2 W^2_{t+\Delta} \end{bmatrix}$$

Chapter 6

(3) $$E_t[S_T] = E[S_T|I_t], \quad t < T$$

(4) $$E\,|S_t| < \infty$$

(5) $$E_t\left[S_T\right] = S_t, \text{ for all } t < T$$

(10) $E_t^{\tilde{P}}[e^{-ru}B_{t+u}] = B_t, \quad 0 < u < T - t$

(11) $E_t^{\tilde{P}}[e^{-ru}S_{t+u}] = S_t, \quad 0 < u$

(22) $\Delta M_t = \Delta N_t^G - \Delta N_t^B$

(27) $P(\Delta N_t^G = 1) \cong \lambda^G \Delta > P(\Delta N_t^B = 1) \cong \lambda^B \Delta$

(28) $E_t[\Delta M_t] \cong \lambda^G \Delta - \lambda^B \Delta > 0$

(41) $\Delta X_t \sim N(\mu\Delta, \sigma^2\Delta)$

(44) $X_{t+T} = X_0 + \int_0^{t+T} dX_u$

(48) $Z_t = X_t - \mu t$

(52) $E_t[Z_{t+T}] = X_t - \mu t$

(53) $E_t[Z_{t+T}] = Z_t$

(54) $\Delta S_t \sim N(0, \sigma^2\Delta)$

(55) $S_0 = 0$

(56) $Z_t = S_t^2$

(58) $E_t[Z_{t+T} - \sigma^2(T + t)] = Z_t - \sigma^2 t$

(62) $I_t \subseteq I_{t+1} \subseteq \ldots \subseteq I_{T-1} \subseteq I_T$

(63) $M_t = E^P[Y_T|I_t]$

(64) $E^P[M_{t+s}|I_t] = M_t$

(69) $G_T = f(S_T)$

(70) $B_T = e^{\int_t^T r_s ds}$

(71) $M_t = E^P\left[\frac{G_T}{B_T}|I_t\right]$

(105) $M_{t_k} = M_{t_0} + \sum_{i=1}^{k} H_{t_{i-1}}[Z_{t_i} - Z_{t_{i-1}}]$

(107) $E_{t_0}[M_{t_k}] = M_{t_0}$

(111) $C_T = C_t + \int_t^T D_s ds + \int_t^T g(C_s) dM_s$

Chapter 7

(24) $\Delta W_k = [S_k - S_{k-1}] - E_{k-1}[S_k - S_{k-1}]$

(27) $W_k = \sum_{i=1}^{k} \Delta W_i$

(29) $E_{k-1}W_k = W_{k-1}$

(30) $V_k = E_0[\Delta W_k^2]$

(31) $V = E_0\left[\sum_{k=1}^{n} \Delta W_k\right]^2 = \sum_{k=1}^{n} V_k$

(32) $V > A_1 > 0$

(34) $V < A_2 < \infty$

(35) $V_{max} = \max_k [V_k, k = 1, \ldots, n]$

(36) $\frac{V_k}{V_{max}} > A_3, \quad 0 < A_3 < 1$

(37) $E[\Delta W_k]^2 = \sigma_k^2 h$

(52) $S_k - S_{k-1} = E_{k-1}[S_k - S_{k-1}] + \sigma_k \Delta W_k$

Chapter 8

(7) $dN_t = \begin{cases} 1 & \text{with probability } \lambda dt \\ 0 & \text{with probability } 1 - \lambda dt \end{cases}$

(8) $M_t = N_t - \lambda t$

(9) $E[M_t] = 0$

(14) $\sigma_k \Delta W_k = \begin{cases} w_1 & \text{with probability } p_1 \\ w_2 & \text{with probability } p_2 \\ \vdots & \qquad \vdots \\ w_m & \text{with probability } p_m \end{cases}$

(15) $E[\sigma_k \Delta W_k]^2 = \sigma_k^2 h$

(17) $\sum_{i=1}^{m} p_i w_i^2 = \sigma_k^2 h$

(22) $w_i(h) = \bar{w}_i h^{r_i}$

(23) $p_i(h) = \bar{p}_i h^{q_i}$

(27) $q_i + 2r_i = 1$

(28) $c_i = \bar{w}_i^2 \bar{p}_i$

(53) $J_t = (N_t - \lambda t)$

(55) $dS_t = a(S_t, t)dt + \sigma_1(S_t, t)dW_t + \sigma_2(S_t, t)dJ_t$

(70) $t_0 = 0 < t_1 < \ldots < t_n = T$

(71) $n\Delta = T$

(72) $S_i = S_{t_i}, \quad i = 0, 1, \ldots, n$

(73) $S_{i+1} = \begin{cases} u_i S_i & \text{with probability } p_i \\ d_i S_i & \text{with probability } 1 - p_i \end{cases}$

(74) $u_i = e^{\sigma\sqrt{\Delta}}, \quad \text{for all } i$

(75) $d_i = e^{-\sigma\sqrt{\Delta}}, \quad \text{for all } i$

(76) $p_i = \frac{1}{2}\left[1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right], \quad \text{for all } i$

(85) $\frac{S_{i+n\Delta}}{S_i}$

$$(86)\qquad \log \frac{S_{i+n}}{S_i} = Z \log u + (n - Z) \log d$$

$$(87)\qquad \log \frac{S_{i+n}}{S_i} = Z \log \frac{u}{d} + n \log d$$

Chapter 9

$$(37)\qquad \lim_{n\to\infty} E \left[\sum_{k=1}^{n} \sigma(S_{k-1}, k) \left[W_k - W_{k-1}\right] - \int_0^T \sigma(S_u, u) dW_u \right]^2 = 0$$

$$(38)\qquad S_k - S_{k-1} = a(S_{k-1}, k)h + \sigma(S_{k-1}, k)[W_k - W_{k-1}], \quad k = 1, 2, \ldots, n$$

$$(39)\qquad E \left[\int_0^T \sigma(S_t, t)^2 dt \right] < \infty$$

$$(41)\qquad \sum_{k=1}^{n} \sigma(S_{k-1}, k)[W_k - W_{k-1}] \to \int_0^T \sigma(S_t, t) dW_t$$

$$(73)\qquad \int_0^T x_t dx_t = \frac{1}{2} \left[x_T^2 - T\right]$$

$$(74)\qquad \lim_{n\to\infty} E \left[\sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2 - T \right]^2 = 0$$

$$(76)\qquad \text{If } \int_0^T (dx_t)^2 \text{ exists, then } \lim_{n\to\infty} E \left[\sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2 - \int_0^T (dx_t)^2 \right]^2 = 0$$

$$(77)\qquad \int_0^T dt = T$$

$$(78)\qquad \int_0^T (dx_t)^2 = \int_0^T dt$$

$$(79)\qquad (dW_t)^2 = dt$$

$$(85)\qquad E_s \left[\int_0^t \sigma_u dW_u \right] = \int_0^s \sigma_u dW_u, \quad 0 < s < t$$

Chapter 10

$$(nn)\qquad dS_t = a_t dt + \sigma_t dW_t, \quad t \geq 0$$

$$(36)\qquad dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma_t^2 dt$$

$$(37)\qquad dF_t = \left[\frac{\partial F}{\partial S_t} a_t + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma_t^2 \right] dt + \frac{\partial F}{\partial S_t} \sigma_t dW_t$$

$$(64)\qquad \int_0^t F_s dS_u = [F(S_t, t) - F(S_0, 0)] - \int_0^t \left[F_u + \frac{1}{2} F_{ss} \sigma_u^2 \right] du$$

(69) $$dF_t = F_t dt + F_{s_1} dS_1 + F_{s_2} dS_2 + \frac{1}{2}\left[F_{s_1 s_1} dS_1^2 + F_{s_2 s_2} dS_2^2 + 2F_{s_1 s_2} dS_1 dS_2\right]$$

(79) $$Y(t) = \sum_{i=1}^{n} N_i(t) P_i(t)$$

(80) $$dY(t) = \sum_{i=1}^{n} N_i(t) dP_i(t) + \sum_{i=1}^{n} dN_i(t) P_i(t) + \sum_{i=1}^{n} dN_i(t) dP_i(t)$$

(81) $$dS_t = a_t dt + \sigma_t dW_t + dJ_t, \quad t \geq 0$$

(82) $$E[\Delta J_t] = 0$$

(83) $$\Delta J_t = \Delta N_t - \left[\lambda_t h \left(\sum_{i=1}^{k} a_i p_i\right)\right]$$

(84) $$a_t = \alpha_t + \lambda_t \left(\sum_{i=1}^{k} a_i p_i\right)$$

(85) $$dF(S_t, t) = \left[F_t + \lambda_t \sum_{i=1}^{k} (F(S_t + a_i, t) - F(S_t, t)) p_i + \frac{1}{2} F_{ss} \sigma^2\right] dt + F_s dS_t + dJ_F$$

(86) $$dJ_F = [F(S_t, t) - F(S_t^-, t)] - \lambda_t \left[\sum_{i=1}^{k} (F(S_t + a_i, t) - F(S_t, t)) p_i\right] dt$$

(87) $$S_t^- = \lim_{s \to t} S_s, \quad s < t$$

Chapter 11

(24) $$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \in [0, \infty)$$

(30) $$S_t = S_0 e^{\{(a - \frac{1}{2}\sigma^2)t + \sigma W_t\}}$$

(34) $$dS_t = r S_t dt + \sigma S_t dW_t, \quad t \in [0, \infty)$$

(38) $$S_T = \left[S_0 e^{(r - \frac{1}{2}\sigma^2)T}\right] \left[e^{\sigma W_T}\right]$$

(42) $$Z_t = e^{\sigma W_t}$$

(50) $$x_t = E[Z_t] = e^{\frac{1}{2}\sigma^2 t}$$

(56) $$S_t = e^{-r(T-t)} E_t[S_T]$$

(72) $$dS_t = \mu S_t dt + \sigma \sqrt{S_t} dW_t, \quad t \in [0, \infty)$$

(74) $$dS_t = \lambda(\mu - S_t) dt + \sigma S_t dW_t$$

(78) $$dS_t = -\mu S_t dt + \sigma dW_t$$

(79) $$dS_t = \mu dt + \sigma_t dW_{1t}$$

(80) $$d\sigma_t = \lambda(\sigma_0 - \sigma_t) dt + \alpha \sigma_t dW_{2t}$$

Chapter 12

Note - Formulas (24), (26), and (28) are incorrect in the text. The correct versions are given here.

(3) $P_t = \theta_1 F(S_t, t) + \theta_2 S_t$

(4) $dP_t = \theta_1 dF_t + \theta_2 dS_t$

(5) $dS_t = a(S_t, t)dt + \sigma(S_t, t)dW_t, \quad t \in [0, \infty)$

(6) $dF_t = F_t dt + \frac{1}{2} F_{ss}\sigma_t^2 dt + F_s dS_t$

(7) $dF_t = \left[F_s a_t + \frac{1}{2} F_{ss}\sigma_t^2 + F_t\right] dt + F_s \sigma_t dW_t$

(10) $\theta_1 = 1$

(11) $\theta_2 = -F_s$

(12) $dP_t = F_t dt + \frac{1}{2} F_{ss}\sigma_t^2 dt$

(16) $r(F(S_t, t) - F_s S_t) = F_t + \frac{1}{2} F_{ss}\sigma_t^2$

(17) $-rF + rF_s S_t + F_t + \frac{1}{2} F_{ss}\sigma_t^2 = 0, \quad 0 \le S_t, \quad 0 \le t \le T$

(23) $P_t = F(S_t, t) - F_s(S_t, t)S_t$

(24) $dP_t = dF(S_t, t) - F_s dS_t - S_t dF_s - dF_s(S_t, t)dS_t$

(26) $dP_t = dF(S_t, t) - F_s dS_t - S_t \left[\left[F_{st} + F_{ss}\mu S_t + \frac{1}{2} F_{sss}\sigma_t^2 S_t^2\right] dt + F_{ss}\sigma S_t dW_t\right] - F_{ss}\sigma_t^2 S_t^2 dt$

(28) $dP_t = dF(S_t, t) - F_s dS_t - S_t[F_{ss}(\mu - r)S_t dt] - F_{ss}\sigma S_t^2 dW_t$

(nn) $dW_t^* = (\sigma dW_t + (\mu - r)dt)$

(29) $a_0 F + a_1 F_s S_t + a_2 F_t + a_3 F_{ss} = 0, \quad 0 \le S_t, \quad 0 \le t \le T$

(30) $F(S_T, T) = G(S_T, T)$

(32) $F_t + F_s = 0, \quad 0 \le S_t, \quad 0 \le t \le T$

(33) $F(S_t, t) = \alpha S_t - \alpha t + \beta$

(44) $F(S_t, t) = e^{\alpha S_t - \alpha t}$

(46) $-.3F_{ss} + F_{tt} = 0$

(47) $F(S_t, t) = \frac{1}{2}\alpha(S_t - S_0)^2 + \frac{.3}{2}\alpha(t - t_0)^2 + \beta(S_t - S_0)(t - t_0)$

(53) $F(S_t, t) = -10(S_t - 4)^2 - 3(t - 2)^2, \quad -10 \le t \le 10, \ -10 \le S_t \le 10$

Chapter 13

(1) $a(S_t, t) = \mu S_t$

(2) $\sigma(S_t, t) = \sigma S_t, \quad t \in [0, \infty)$

(3) $-rF + rF_s S_t + F_t + \frac{1}{2} F_{ss}\sigma^2 S_t^2 = 0, \quad 0 \le S_t, \quad 0 \le t \le T$

(4) $F(T) = \max[S_T - K, 0]$

(6) $F(S_t, t) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2)$

(7) $$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

(8) $$d_2 = d_1 - \sigma\sqrt{T-t}$$

(9) $$N(d_1) = \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

(17) $$rF - rF_s S_t + \delta F_s S_t - F_t - \frac{1}{2} F_{ss}\sigma_t^2 = 0$$

(18) $F(S_{1T}, S_{2T}, T) = \max[0, \max(S_{1T}, S_{2T}) - K]$ (multi-asset option)

(19) $F(S_{1T}, S_{2T}, T) = \max[0, (S_{1T} - S_{2T}) - K]$ (spread call option)

(20) $F(S_{1T}, S_{2T}, T) = \max[0, (\theta_1 S_{1T} + \theta_2 S_{2T}) - K]$ (portfolio call option)

(21) $F(S_{1T}, S_{2T}, T) = \max[0, (S_{1T} - K_1), (S_{2T} - K_2)]$ (dual strike option)

(30) $$\frac{\Delta F}{\Delta t} + rS\frac{\Delta F}{\Delta S} + \frac{1}{2}\sigma^2 S^2 \frac{\Delta^2 F}{\Delta S^2} \cong rF$$

(31) $$\frac{\Delta F}{\Delta t} \cong \frac{F_{ij} - F_{i,j-1}}{\Delta t}$$

(32) $$rS\frac{\Delta F}{\Delta S} \cong rS_j \frac{F_{ij} - F_{i-1,j}}{\Delta S}$$

(33) $$rS\frac{\Delta F}{\Delta S} \cong rS_j \frac{F_{i+1,j} - F_{ij}}{\Delta S}$$

(34) $$\frac{\Delta^2 F}{\Delta S^2} \cong \left[\frac{F_{i+1,j} - F_{ij}}{\Delta S} - \frac{F_{ij} - F_{i-1,j}}{\Delta S}\right]\frac{1}{\Delta S}$$

Chapter 14

(6) $$dP(\bar{z}) = P\left(\bar{z} - \frac{1}{2}dz_t < z_t < \bar{z} + \frac{1}{2}dz_t\right)$$

(7) $$\int_{-\infty}^{\infty} dP(z_t) = 1$$

(8) $$E[z_t] = \int_{-\infty}^{\infty} z_t dP(z_t)$$

(9) $$E\left[z_t - E[z_t]\right]^2 = \int_{-\infty}^{\infty} \left[z_t - E[z_t]\right]^2 dP(z_t)$$

(29) $$E_t\left[\frac{1}{(1+R_t)} S_{t+1}\right] = S_t$$

(31) $$E_t^{\tilde{P}}\left[\frac{1}{(1+r_t)} S_{t+1}\right] = S_t$$

(41) $$z_t \sim N(0,1)$$

(42) $$dP(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_t)^2} dz_t$$

(43) $$\xi(z_t) = e^{z_t\mu - \frac{1}{2}\mu^2}$$

(44) $$[dP(z_t)][\xi(z_t)] = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_t^2) + \mu z_t - \frac{1}{2}\mu^2} dz_t$$

(45) $$d\tilde{P}(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[z_t - \mu]^2} dz_t$$

(46) $d\tilde{P}(z_t) = dP(z_t)\xi(z_t)$

(48) $\xi(z_t)^{-1} d\tilde{P}(z_t) = dP(z_t)$

(53) $$f(z_{1t}, z_{2t}) = \frac{1}{2\pi\sqrt{|\Omega|}} e^{-\frac{1}{2}\left[\begin{array}{cc}(z_{1t}-\mu_1) & (z_{2t}-\mu_2)\end{array}\right]\left[\begin{array}{cc}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{array}\right]^{-1}\left[\begin{array}{c}(z_{1t}-\mu_1)\\ (z_{2t}-\mu_2)\end{array}\right]}$$

(54) $$\Omega = \left[\begin{array}{cc}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{array}\right]$$

(55) $|\Omega| = \sigma_1^2\sigma_2^2 - \sigma_{12}^2$

(56) $dP(z_{1t}, z_{2t}) = f(z_{1t}, z_{2t})dz_{1t}dz_{2t}$

(57) $$\xi(z_{1t}, z_{2t}) = \exp\left\{-\left[\begin{array}{cc}z_{1t} & z_{2t}\end{array}\right]\left[\begin{array}{cc}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{array}\right]^{-1}\left[\begin{array}{c}\mu_1\\ \mu_2\end{array}\right] + \frac{1}{2}\left[\begin{array}{cc}\mu_1 & \mu_2\end{array}\right]\left[\begin{array}{cc}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{array}\right]^{-1}\left[\begin{array}{c}\mu_1\\ \mu_2\end{array}\right]\right\}$$

(58) $d\tilde{P}(z_{1t}, z_{2t}) = \xi(z_{1t}, z_{2t})dP(z_{1t}, z_{2t})$

(59) $$d\tilde{P}(z_{1t}, z_{2t}) = \left[\frac{1}{2\pi\sqrt{|\Omega|}} e^{-\frac{1}{2}\left[\begin{array}{cc}z_{1t} & z_{2t}\end{array}\right]\left[\begin{array}{cc}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{array}\right]^{-1}\left[\begin{array}{c}z_{1t}\\ z_{2t}\end{array}\right]}\right] dz_{1t}dz_{2t}$$

(60) $\xi(z_t) = e^{-z_t'\Omega^{-1}\mu + \frac{1}{2}\mu'\Omega^{-1}\mu}$

(69) $$\frac{d\tilde{P}(z_t)}{dP(z_t)} = \xi(z_t)$$

(74) $d\tilde{P}(z_t) = \xi(z_t)dP(z_t)$

(75) $dP(z_t) = \xi(z_t)^{-1}d\tilde{P}(z_t)$

(76) $\xi_t = e^{\left(\int_0^t X_u dW_u - \frac{1}{2}\int_0^t X_u^2 du\right)}, \quad t \in [0, T]$

(77) $E\left[e^{\int_0^t X_u^2 du}\right] < \infty, \quad t \in [0, T]$

(83) $$E\left[\int_0^t \xi_s X_s dW_s | I_u\right] = \int_0^u \xi_s X_s dW_s$$

(84) $$\tilde{W}_t = W_t - \int_0^t X_u du, \quad t \in [0, T]$$

(85) $\tilde{P}_T(A) = E^P[1_A \xi_T]$

(86) $d\tilde{W}_t = dW_t - X_t dt$

(93) $d\tilde{P}_T = \xi_T dP$

(122) $A_1 + A_2 + \cdots + A_n = \Omega$

(123) $1_{A_1} + 1_{A_2} + \ldots 1_{A_n} = 1 = 1_\Omega$

(127) $E^P[Z_t 1_{A_i}] = \tilde{P}(A_i)$

(138) $$E^P[g(X_t)] = \int_\Omega g(x)f(x)dx$$

(*nn*) $g(X_t) = Z_t h(X_t)$

(*nn*) $$E^P[g(X_t)] = \int_\Omega h(x)\tilde{f}(x)dx = E^{\tilde{P}}[h(x)]$$

Chapter 15

(2) $Y_t \sim N(\mu t, \sigma^2 t)$

(4) $M(\lambda) = E[e^{Y_t \lambda}]$

(10) $M(\lambda) = e^{\lambda\mu t + \frac{1}{2}\sigma^2 t\lambda^2}$

(15) $S_t = S_0 e^{Y_t} \quad , t \in [0, \infty)$

(25) $E[S_t|S_u, u < t] = S_u e^{\mu(t-s)+\frac{1}{2}\sigma^2(t-s)}$

(30) $Z_t = e^{-rt} S_t$

(31) $E^{\tilde{P}}\left[e^{-rt} S_t | S_u, u < t\right] = e^{-ru} S_u$

(32) $E^{\tilde{P}}[Z_t|Z_u, u < t] = Z_u$

(38) $E^{\tilde{P}}\left[e^{-r(t-u)} S_t | S_u, u < t\right] = S_u e^{-r(t-u)} e^{\rho(t-u)+\frac{1}{2}\sigma^2(t-u)}$ where $\tilde{P} \sim N(\rho t, \sigma^2 t)$

(42) $E^{\tilde{P}}\left[e^{-rt} S_t | S_u, u < t\right] = e^{-ru} S_u$

(51) $dS_t = rS_t dt + \sigma S_t d\tilde{W}_t$

(58) $C_t = E_t^{\tilde{P}}\left[e^{-r(T-t)} \max\{S_T - K, 0\}\right]$

(88) $dS_t = \mu_t dt + \sigma_t dW_t$

(90) $d[e^{-rt} S_t] = e^{-rt}[\mu_t - rS_t]dt + e^{-rt}\sigma_t dW_t$

(92) $d\tilde{W}_t = dX_t + dW_t$

(97) $dX_t = \left[\frac{\mu_t - rS_t}{\sigma_t}\right] dt$

(98) $d[e^{-rt} S_t] = e^{-rt}\sigma_t d\tilde{W}_t$

(111) $d\left[e^{-rt} F(S_t, t)\right] = e^{-rt}\sigma_t F_s d\tilde{W}_t$

Chapter 17

(6) $$\begin{bmatrix} 1 \\ 0 \\ B^s_{t_1} \\ B_{t_1} \\ C_{t_1} \end{bmatrix} = \begin{bmatrix} R_{t_1} R^u_{t_2} & R_{t_1} R^u_{t_2} & R_{t_1} R^d_{t_2} & R_{t_1} R^d_{t_2} \\ (F_{t_1} - L^u_{t_2}) & (F_{t_1} - L^u_{t_2}) & (F_{t_1} - L^d_{t_2}) & (F_{t_1} - L^d_{t_2}) \\ 1 & 1 & 1 & 1 \\ B^{uu}_{t_3} & B^{ud}_{t_3} & B^{du}_{t_3} & B^{dd}_{t_3} \\ C^{uu}_{t_3} & C^{ud}_{t_3} & C^{du}_{t_3} & C^{dd}_{t_3} \end{bmatrix} \begin{bmatrix} \psi^{uu} \\ \psi^{ud} \\ \psi^{du} \\ \psi^{dd} \end{bmatrix}$$

(13) $1 = R_{t_1} R^u_{t_2} \psi^{uu} + R_{t_1} R^u_{t_2} \psi^{ud} + R_{t_1} R^d_{t_2} \psi^{du} + R_{t_1} R^d_{t_2} \psi^{dd}$

(14) $\tilde{P}^{ij} = (1 + r_{t_1})(1 + r^i_{t_2})\psi^{ij}$

(15) $1 = \tilde{P}^{uu} + \tilde{P}^{ud} + \tilde{P}^{du} + \tilde{P}^{dd}$

(16) $\tilde{P}^{ij} > 0$

(18) $B^s_{t_1} = E^{\tilde{P}}\left[\frac{1}{(1 + r_{t_1})(1 + r_{t_2})}\right]$

(21) $B_{t_1} = E^{\tilde{P}}\left[\frac{B_{t_3}}{(1 + r_{t_1})(1 + r_{t_2})}\right]$

(22) $0 = E^{\tilde{P}}\left[\frac{1}{(1 + r_{t_1})(1 + r_{t_2})}[F_{t_1} - L_{t_2}]\right]$

$$(23)\qquad C_{t_1} = E^{\tilde{P}}\left[\frac{1}{(1+r_{t_1})(1+r_{t_2})}C_{t_3}\right]$$

$$(31)\qquad F_{t_1} = \frac{1}{E^{\tilde{P}}\left[\frac{1}{(1+r_{t_1})(1+r_{t_2})}\right]}E^{\tilde{P}}\left[\frac{1}{(1+r_{t_1})(1+r_{t_2})}L_{t_2}\right]$$

$$(36)\qquad B^s_{t_1} = \psi^{uu} + \psi^{ud} + \psi^{du} + \psi^{dd}$$

$$(38)\qquad \pi^{ij} = \frac{1}{B^s_{t_1}}\psi^{ij}$$

$$(39)\qquad 1 = \pi^{uu} + \pi^{ud} + \pi^{du} + \pi^{dd}$$

$$(46)\qquad F_{t_1} = E^{\pi}[L_{t_2}]$$

$$(52)\qquad C_{t_3} = N\max[L_{t_2} - K, 0]$$

$$(53)\qquad C_{t_1} = E^{\tilde{P}}\left[\frac{1}{(1+r_{t_1})(1+r_{t_2})}\max[L_{t_2} - K, 0]\right]$$

$$(55)\qquad C_{t_1} = B^s_{t_1}E^{\pi}\max[L_{t_2} - K, 0]$$

$$(56)\qquad V_t = E^{\tilde{P}}_t\left[e^{-\int_t^{T+\delta} r_u du}(F_t - L_T)N\delta\right]$$

$$(61)\qquad V_t = E^{\pi}_t[B(t, T+\delta)(F_t - L_T)N\delta]$$

$$(62)\qquad V_t = B(t, T+\delta)E^{\pi}_t[(F_t - L_T)N\delta]$$

$$(63)\qquad F_t = E^{\pi}_t[L_T]$$

Chapter 18

$$(3)\qquad B(t, T) = e^{-R(t,T)(T-t)}, \quad t < T$$

$$(12)\qquad B(t, T) = E^{\tilde{P}}_t\left[e^{-\int_t^T r_s ds}\right]$$

$$(20)\qquad R(t, T) = \frac{-\log E^{\tilde{P}}_t\left[e^{-\int_t^T r_s ds}\right]}{T - t}$$

$$(33)\qquad B(t, T) = e^{-\int_t^T F(t,s)ds}$$

$$(39)\qquad F(t, T) = \lim_{\Delta\to 0}\frac{\log B(t, T) - \log B(t, T+\Delta)}{\Delta}$$

$$(40)\qquad F(t, T, U) = \frac{\log B(t, T) - \log B(t, U)}{U - T}$$

Chapter 19

Note - Formula (21) is incorrect in the text. The correct version (with subtraction rather than addition) is given here.

$$(14)\qquad dB_t = \mu(t, T, B_t)B_t dt + \sigma(t, T, B_t)B_t dV^T_t$$

$$(15)\qquad dB_t = r_t B_t dt + \sigma(t, T, B_t)B_t dW^T_t$$

$$(21)\qquad dF(t, T) = \sigma(t, T, B(t, T))\left[\frac{\partial\sigma(t, T, B(t, T))}{\partial T}\right]dt - \left[\frac{\partial\sigma(t, T, B(t, T))}{\partial T}\right]dW_t$$

$$(22)\qquad dF(t, T) = a(F(t, T), t)dt + b(F(t, T), t)dW_t$$

$$(25)\qquad r_t = F(t, t)$$

(nn) $F(t,T) = F(0,T) + \int_0^t b(s,T)\left[\int_s^T b(s,u)du\right] ds + \int_0^t b(s,T)dW_s$

(26) $r_t = F(0,t) + \int_0^t b(s,t)\left[\int_s^t b(s,u)du\right] ds + \int_0^t b(s,t)dW_s$

(33) $dF(t,T) = b^2(T-t)dt + bdW_t$

(34) $dB(t,T) = r_t B(t,T)dt + b(T-t)B(t,T)dW_t$

(35) $r_t = F(0,t) + \frac{1}{2}b^2t^2 + bW_t$

(36) $dr_t = (F_t(0,t) + b^2t)dt + bdW_t$

(37) $F_t(0,t) = \frac{\partial F(0,t)}{\partial t}$

Paul Wilmott Introduces Quantitative Finance, 2nd Edition, P. Wilmott

Chapter 6

(6.6) $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$

(6.7) $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0$

(6.8) $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-r_f)S\frac{\partial V}{\partial S} - rV = 0$

(6.9) $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r+q)S\frac{\partial V}{\partial S} - rV = 0$

(6.10) $\frac{\partial \mathcal{V}}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 \mathcal{V}}{\partial F^2} + -r\mathcal{V} = 0$

Chapter 8

(8.7) $V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty e^{-(\log(S/S')+(r-\frac{1}{2}\sigma^2)(T-t))^2/2\sigma^2(T-t)}\text{Payoff}(S')\frac{dS'}{S'}$

Call option value

$$Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\log(S/E)+(r-D+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/E)+(r-D-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

Put option value

$$-Se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2)$$

Binary call option value

$$e^{-r(T-t)}N(d_2)$$

Binary put option value

$$e^{-r(T-t)}(1-N(d_2))$$

Delta of common contracts	
Call	$e^{-D(T-t)}N(d_1)$
Put	$e^{-D(T-t)}(N(d_1)-1)$
Binary call	$\frac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$
Binary put	$-\frac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$
	$N'(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$

Gamma of common contracts	
Call	$\frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Put	$\frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Binary call	$-\frac{e^{-r(T-t)}d_1N'(d_2)}{\sigma^2S^2(T-t)}$
Binary Put	$\frac{e^{-r(T-t)}d_1N'(d_2)}{\sigma^2S^2(T-t)}$

Thetas of common contracts	
Call	$-\frac{\sigma Se^{-D(T-t)}N'(d_1)}{2\sqrt{T-t}}+DSN(d_1)e^{-D(T-t)}-rEe^{-r(T-t)}N(d_2)$
Put	$-\frac{\sigma Se^{-D(T-t)}N'(-d_1)}{2\sqrt{T-t}}-DSN(-d_1)e^{-D(T-t)}+rEe^{-r(T-t)}N(-d_2)$
Binary call	$re^{-r(T-t)}N(d_2)+e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)}-\frac{r-D}{\sigma\sqrt{T-t}}\right)$
Binary put	$re^{-r(T-t)}(1-N(d_2))-e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)}-\frac{r-D}{\sigma\sqrt{T-t}}\right)$

Speed of common contracts	
Call	$-\frac{e^{-D(T-t)}N'(d_1)}{\sigma^2S^2(T-t)}\left(d_1+\sigma\sqrt{T-t}\right)$
Put	$-\frac{e^{-D(T-t)}N'(d_1)}{\sigma^2S^2(T-t)}\left(d_1+\sigma\sqrt{T-t}\right)$
Binary call	$-\frac{e^{-r(T-t)}N'(d_2)}{\sigma^2S^3(T-t)}\left(-2d_1+\frac{1-d_1d_2}{\sigma\sqrt{T-t}}\right)$
Binary put	$\frac{e^{-r(T-t)}N'(d_2)}{\sigma^2S^3(T-t)}\left(-2d_1+\frac{1-d_1d_2}{\sigma\sqrt{T-t}}\right)$

Vegas of common contracts	
Call	$S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$
Put	$S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$
Binary call	$-e^{-r(T-t)}N'(d_2)\frac{d_1}{\sigma}$
Binary put	$e^{-r(T-t)}N'(d_2)\frac{d_1}{\sigma}$

Rhos of common contracts	
Call	$E(T-t)e^{-r(T-t)}N(d_2)$
Put	$-E(T-t)e^{-r(T-t)}N(-d_2)$
Binary call	$-(T-t)e^{-r(T-t)}N(d_2)+\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$
Binary put	$-(T-t)e^{-r(T-t)}(1-N(d_2))-\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

Sensitivity to dividend for common contracts	
Call	$-(T-t)Se^{-D(T-t)}N(d_1)$
Put	$(T-t)Se^{-D(T-t)}N(-d_1)$
Binary call	$-\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$
Binary put	$\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

Chapter 9

All formulas are in Section 9.3

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N} R_i^2$$

$$R_i = \frac{S_i - S_{i-1}}{S_{i-1}}$$

$$\sigma_n^2 = \alpha\bar{\sigma}^2 + (1-\alpha)\frac{1}{n}\sum_{i=1}^{n} R_i^2$$

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)R_n^2$$

$$\sigma_n^2 = \alpha\bar{\sigma}^2 + (1-\alpha)(\lambda\sigma_{n-1}^2 + (1-\lambda)R_n^2)$$

$$E[\sigma_{n+k}^2] = E[\sigma_{n+k-1}^2]$$

$$E[\sigma_{n+k}^2] = \bar{\sigma}^2 + \left(E[\sigma_n^2] - \bar{\sigma}^2\right)(1-\nu)^k$$

$$\nu = \frac{\alpha}{1-(1-\alpha)(1-\lambda)}$$

$$\sigma_{CC}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\log\left(\frac{C_i}{C_{i-1}}\right)\right)^2$$

Parkinson: $\sigma_p^2 = \frac{1}{4n\log(2)}\sum_{i=1}^{n}\left(\log\left(\frac{H_i}{L_i}\right)\right)^2$

Garman & Klass: $\sigma_{gk}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(0.511\left(\log\left(\frac{H_i}{L_i}\right)\right)^2 - 0.019\log\left(\frac{C_i}{O_i}\right)\log\left(\frac{H_iL_i}{O_i^2}\right) - 2\log\left(\frac{H_i}{O_i}\right)\log\left(\frac{L_i}{O_i}\right)\right)$

Rogers & Satchell: $\sigma_{rs}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\log\left(\frac{H_i}{C_i}\right)\log\left(\frac{H_i}{O_i}\right) + \log\left(\frac{L_i}{C_i}\right)\log\left(\frac{L_i}{O_i}\right)\right)$

Chapter 10

Section 10.4, one time step mark-to-market profit (using Black-Scholes with $\sigma = \tilde{\sigma}$)

$$= \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)S^2\Gamma^i dt + (\Delta^i - \Delta^a)((\mu - r + D)S dt + \sigma S dX)$$

Section 10.5, mark-to-market profit from today to tomorrow

$$= \frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)S^2\Gamma^i dt$$

Chapter 14

(14.1) $$V = Pe^{-y(T-t)} + \sum_{i=1}^{N} C_i e^{-y(t_i - t)}$$

Chapter 16

(16.4) $$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0$$

(16.5) $$dV - rV\,dt = w\frac{\partial V}{\partial r}(dX + \lambda dt)$$

16.6 NAMED MODELS

16.6.1 Vasicek

$$dr = (\eta - \gamma r)dt + \beta^{1/2} dX$$

The value of a zero coupon bond is given by $e^{A(t;T) - rB(t;T)}$

$$B = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})$$

$$A = \frac{1}{\gamma^2}(B(t;T) - T + t)(\eta\gamma - \frac{1}{2}\beta) - \frac{\beta B(t;T)^2}{4\gamma}$$

16.6.2 Cox, Ingersoll & Ross

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}\, dX$$

The value of a zero coupon bond is given by $e^{A(t;T) - rB(t;T)}$

$$\frac{\alpha}{2} A = a\psi_2 \log(a - B) + \psi_2 b \log((B + b)/b) - a\psi_2 \log a$$

$$B(t;T) = \frac{2(e^{\psi_1(T-t)} - 1)}{(\gamma + \psi_1)(e^{\psi_1(T-t)} - 1) + 2\psi_1}$$

$$\psi_1 = \sqrt{\gamma^2 + 2\alpha} \text{ and } \psi_2 = \frac{\eta}{a + b}$$

$$b, a = \frac{\pm\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha}$$

16.6.3 Ho & Lee

$$dr = \eta(t)dt + \beta^{1/2} dX$$

The value of a zero coupon bond is given by $e^{A(t;T) - rB(t;T)}$

$$B = T - t$$

$$A = -\int_t^T \eta(s)(T - s)ds + \frac{1}{6}\beta(T - t)^3$$

$$\eta(t) = -\frac{\partial^2}{\partial t^2} \log Z_M(t^*; t) + \beta(t - t^*)$$

Section 16.7

$$\text{Forward price} = \frac{S}{Z}$$

$$\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} + (u - \lambda w)\frac{\partial Z}{\partial r} - rZ = 0$$

with $Z(r, T) = 1$

Section 16.8

$$\text{Futures price } F(S, r, t) = \frac{S}{p(r, t)}$$

$$\frac{\partial p}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 p}{\partial r^2} + (\mu - \lambda w)\frac{\partial p}{\partial r} - rp - w^2 \frac{\left(\frac{\partial p}{\partial r}\right)^2}{q} + p\sigma\beta\frac{\partial p}{\partial r} = 0$$

with $p(r, T) = 1$

Chapter 17

Section 17.2 Ho & Lee

$$dr = \eta(t)dt + cdX$$

$$Z(r, t : T) = e^{A(t;T) - r(T-t)}$$

$$A(t;T) = \log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right) - (T-t)\frac{\partial}{\partial t}\log(Z_M(t^*;t)) - \frac{1}{2}c^2(t-t^*)(T-t)^2$$

Section 17.3 The Extended Vasicek Model of Hull & White

$$dr = (\eta - \gamma r)dt + cdX$$

$$dr = (\eta(t) - \gamma r)dt + cdX$$

$$Z(r, t : T) = e^{A(t;T) - rB(t;T)}$$

$$B(t;T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right)$$

$$A(t;T) = \log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right) - B(t;T)\frac{\partial}{\partial t}\log(Z_M(t^*;t)) - \frac{c^2}{4\gamma^3}\left(e^{-\gamma(T-t^*)} - e^{-\gamma(t-t^*)}\right)^2\left(e^{2\gamma(t-t^*)} - 1\right)$$

Chapter 18

Section 18.3.1 Bond Options - Market Practice

Call option price: $e^{-r(T-t)}(FN(d_1') - EN(d_2'))$

Put option price: $e^{-r(T-t)}(EN(-d_2') - FN(-d_1'))$

$$d_1' = \frac{\log(F/E) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2' = d_1' - \sigma\sqrt{T-t}$$

Section 18.4.3 Caps and Floors - Market Practice

Caplet price: $e^{-r^*(t_i-t)}(F(t, t_{i-1}, t_i)N(d_1') - r_cN(d_2'))$

Floorlet price: $e^{-r^*(t_i-t)}(-F(t, t_{i-1}, t_i)N(-d_1') + r_cN(-d_2'))$

$$d_1' = \frac{\log(F/r_c) + \frac{1}{2}\sigma^2 t_{i-1}}{\sigma\sqrt{t_{i-1}}}$$

$$d_2' = d_1' - \sigma\sqrt{t_{i-1}}$$

Section 18.6.1 Swaptions, Captions and Floortions - Market Practice

Price of a payer swaption: $\frac{1}{F}e^{-r(T-t)}\left(1 - \left(1 + \frac{1}{2}F\right)^{-2(T_s-T)}\right)(FN(d_1') - EN(d_2'))$

Price of a receiver swaption: $\frac{1}{F}e^{-r(T-t)}\left(1 - \left(1 + \frac{1}{2}F\right)^{-2(T_s-T)}\right)(EN(-d_2') - FN(-d_1'))$

$$d_1' = \frac{\log(F/E) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2' = d_1' - \sigma\sqrt{T-t}$$

Chapter 19

(19.1) $Z(t;T) = e^{-\int_t^T F(t;s)ds}$

(19.2) $dZ(t;T) = \mu(t,T)Z(t;T)dt + \sigma(t,T)Z(t;T)dX$

(nn) $F(t;T) = -\frac{\partial}{\partial T}\log Z(t;T)$

(19.3) $dF(t;T) = \frac{\partial}{\partial T}\left(\frac{1}{2}\sigma^2(t,T) - \mu(t,T)\right)dt - \frac{\partial}{\partial T}\sigma(t,T)dX$

(19.5) $dF(t;T) = \nu(t,T)\left(\int_t^T \nu(t,s)ds\right)dt + \nu(t,T)dX$

(19.6) $dF(t;T) = \left(\sum_{i=1}^{N}\nu_i(t,T)\int_t^T \nu_i(t,s)ds\right)dt + \sum_{i=1}^{N}\nu_i(t,T)dX_i$

(19.7) $dZ_i = rZ_i dt + Z_i\sum_{j=1}^{i-1}a_{ij}dX_j$

(19.8) $dZ_i = (1+\tau F_i)dZ_{i+1} + \tau Z_{i+1}dF_i + \tau\sigma_i F_i Z_{i+1}\left(\sum_{j=1}^{i}a_{i+1,j}\rho_{ij}\right)dt$

(19.9) $dF_i = \left(\sum_{j=1}^{i}\frac{\sigma_j F_j \tau\rho_{ij}}{1+\tau F_j}\right)\sigma_i F_i dt + \sigma_i F_i dX_i$

FAQ's in Option Pricing Theory, P. Carr

(38) $\beta_T = [V(S_0,0;\sigma_i) - V(S_0,0;\sigma_h)]e^{rT} + S_T f'(S_T) - f(S_T) + \int_0^T e^{r(T-t)}(\sigma_t^2 - \sigma_h^2)\frac{S_t^2}{2}\frac{\partial^2}{\partial S^2}V(S_t,t;\sigma_h)dt$

(39) $N_T = f'(S_T)$

(40) $P\&L_T \equiv N_T S_T - \beta_T - f(S_T) = [V(S_0,0;\sigma_i) - V(S_0,0;\sigma_h)]e^{rT} + \int_0^T e^{r(T-t)}(\sigma_h^2 - \sigma_t^2)\frac{S_t^2}{2}\frac{\partial^2}{\partial S^2}V(S_t,t;\sigma_h)dt$

The Handbook of Fixed Income Securities, 8th ed., F. Fabozzi

Page 202 $Y_d = \frac{(F-P)}{F}\times\frac{360}{t}$

Managing Investment Portfolios, a dynamic process, Maginn, et al Chapter 5

(5-1) $U_m = E(R_m) - 0.005R_A\sigma_m^2$

(5-2) $\text{SFRatio} = \frac{E(R_P) - R_L}{\sigma_P}$

(5-3) $\frac{E(R_{new}) - R_F}{\sigma_{new}} > \left(\frac{E(R_p) - R_F}{\sigma_p}\right)\text{Corr}(R_{new}, R_p)$

(5-4) $U_m^{\text{ALM}} = E(\text{SR}_m) - 0.005R_A\sigma^2(\text{SR}_m)$

Quantitative Financial Economics: Stocks, Bonds and Foreign Exchange, K. Cuthbertson and K. Nitzsche (QFIC 101-13)

Chapter 24

(1) $P = SP^* \text{ or } p = s + p^*$

(2) $\Delta p = \Delta s + \Delta p^*$

(3) $S^r = P^*S/P$

(4a) $s = p - p^*$

(4b) $\Delta s = \Delta p - \Delta p^*$

(5) $\Delta p = \dfrac{b_1}{1-b_1}\left[f + b_1(\chi_w - \chi_p) + a_2(y - \bar{y})\right] + \Delta(p+s)$

(6) $f + b_1(\chi_w - \chi_p) + a_2(y - \bar{y}) = 0$

(7) $A(1+r) = (A/S)(1+r^*)F$

(8) $F/S = (1+r)/(1+r^*)$

(9) $f - s = r - r^*$

(10) $S^{\mathrm{e}}_{t+1}/S_t = (1+r_t)/(1+r^*_t)$

(11) $s^{\mathrm{e}}_{t+1} - s_t = r_t - r^*_t$

(12) $1 + E_t R_{t+1}(\mathrm{UK} \rightarrow \mathrm{US}) \equiv S^{\mathrm{e}}_{t+1}(1+r^*_t)/S_t$

(13) $E_t R_{t+1} - r_t = \beta_i(E_t R_{m,t+1} - r_t)$

(14) $f_t = E_t s_{t+1}$

(15) $\Delta s^{\mathrm{e}}_{t+1} = s^{\mathrm{e}}_{t+1} - s_t = (r - r^*)_t$

(16) $\Delta s^{\mathrm{e}}_{t+1} = \Delta p^{\mathrm{e}}_{t+1} - \Delta p^{*\mathrm{e}}_{t+1}$

(17) $r_t - \Delta p^{\mathrm{e}}_{t+1} = r^*_t - \Delta p^{*\mathrm{e}}_{t+1}$

Modern Investment Management: An Equilibrium Approach, B. Litterman (QFIC 106-13)

Chapter 10

(10.1) $R_{L,t} - R_{f,t} = \beta(R_{B,t} - R_{f,t}) + \varepsilon_t$

(10.2) $SR_i = \dfrac{\mu_i - R_f}{\sigma_i}$

(10.3) $S_t \equiv A_t - L_t$

(10.4) $F_t \equiv A_t/L_t$

(10.6) $RACS_t = \dfrac{E_t\left[A_t\left(1+R_{A,t+1}\right) - L_t\left(1+R_{L,t+1}\right) - \left(A_t - L_t\right)\left(1+R_f\right)\right]}{\sigma_t\left[A_t\left(1+R_{A,t+1}\right) - L_t\left(1+R_{L,t+1}\right)\right]}$

(10.8) $\dfrac{\sigma_t\left[S_{t+1}\right]}{A_t}$

(10.9) $\dfrac{\left(1 - \beta\dfrac{L_t}{A_t}\right)\left(\sigma_B^2 - \rho\sigma_B\sigma_E\right)}{\sigma_E^2 + \sigma_B^2 - 2\rho\sigma_B\sigma_E}$

(10.10) $$\frac{\mu_B \left(\beta \frac{L_t}{A_t} - 1\right) + \frac{L_t}{A_t} R_f (1-\beta)}{\mu_E - \mu_B}$$

(10.11) $$A_{t+1} = A_t \left(1 + R_{A,t+1}\right) - pL_t \left(1 + R_{L,t+1}\right) \text{ and } L_{t+1} = L_t \left(1 + R_{L,t+1}\right)(1-p)$$

(10.13) $$E_t \left[R_{x,t+1}\right] = \frac{E\left[F_{t+1}\right](1-p) + p}{F_t}$$

(10.14) $$E_0\left[F_t\right] = \left[\frac{1 + E\left[R_x\right]}{1-p}\right]^t F_0 + p \frac{1 - \left[\frac{1+E\left[R_x\right]}{1-p}\right]^t}{E\left[R_x\right] + p}$$

(10.A.1) $$R_{L,t} - R_{f,t} = \beta\left(R_{B,t} - R_{f,t}\right) + \varepsilon_t$$

(10.A.2) $$V = Ce^{-r(T-t)}$$

(10.A.4) $$\frac{dV}{V} = \frac{dC}{C} - (T-t)dr + rdt$$

(10.A.14) $$\alpha = \frac{\left(1 - \beta \frac{L_t}{A_t}\right)\left(\sigma_B^2 - \rho\sigma_E\sigma_B\right)}{\sigma_E^2 + \sigma_B^2 - 2\rho\sigma_E\sigma_B}$$

(10.A.16) $$\alpha = \frac{\mu_B \left(\beta \frac{L_t}{A_t} - 1\right) + \frac{L_t}{A_t}\left[R_f(1-\beta) + \eta\right]}{\mu_E - \mu_B}$$

(10.A.22) $$E_0\left[F_t\right] = \left(\frac{1+\mu_x}{1-p}\right)^t F_0 + p\frac{1 - \left(\frac{1+\mu_x}{1-p}\right)^t}{\mu_x + p}$$

Analysis of Financial Time Series, third edition, R. Tsay (QFIC 100-13) Chapter 3

(3.2) $$\mu_t = E(r_t|F_{t-1}), \quad \sigma_t^2 = Var(r_t|F_{t-1}) = E[(r_t - \mu_t)^2|F_{t-1}]$$

(3.3) $$r_t = \mu_t + a_t, \quad \mu_t = \sum_{i=1}^{p} \phi_i y_{t-i} - \sum_{i=1}^{q} \theta_i a_{t-i}, \quad y_t = r_t - \phi_0 - \sum_{i=1}^{k} \beta_i x_{it}$$

(3.4) $$\sigma_t^2 = Var(r_t|F_{t-1}) = Var(a_t|F_{t-1})$$

(p114) $$a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2 + e_t. \quad t = m+1, \ldots, T$$

(p114) $$F = \frac{(SSR_0 - SSR_1)/m}{SSR_1/(T-2m-1)}$$

(3.5) $$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2$$

(p117) $$a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2$$

(p118) $$E(a_t^4) = \frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}$$

(p120) $$\begin{aligned} f(a_1, \ldots, a_T|\alpha) &= f(a_T|F_{T-1}) f(a_{T-1}|F_{T-2}) \cdots f(a_{m+1}|F_m) f(a_1, \ldots, a_m|\alpha) \\ &= \prod_{t=m+1}^{T} \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{a_t^2}{2\sigma_t^2}\right) \times f(a_1, \ldots, a_m|\alpha) \end{aligned}$$

(p121) $\ell(a_{m+1},\ldots,a_T|\alpha,\alpha_1,\ldots,\alpha_m) = -\sum_{t=m+1}^{T}\left[\frac{1}{2}\ln(\sigma_t^2) + \frac{1}{2}\frac{a_t^2}{\sigma_t^2}\right]$

(3.7) $f(\epsilon_t|v) = \frac{\Gamma[(v+1)/2]}{\Gamma(v/2)\sqrt{(v-2)\pi}}\left(1+\frac{\epsilon_t^2}{v-2}\right)^{-(v+1)/2}, \; v > 2$

(3.8) $\ell(a_{m+1},\ldots,a_T|\alpha,A_m) = -\sum_{t=m+1}^{T}\left[\frac{v+1}{2}\ln\left(1+\frac{a_t^2}{(v-2)\sigma_t^2}\right) + \frac{1}{2}\ln(\sigma_t^2)\right]$

(p121) $\ell(a_{m+1},\ldots,a_T|\alpha,v,A_m) = (T-m)\left\{\ln\left[\Gamma\left(\frac{v+1}{2}\right)\right] - \ln\left[\Gamma\left(\frac{v}{2}\right)\right]\right.$

$\left. - 0.5\ln[(v-2)\pi]\right\} + \ell(a_{m+1},\ldots,a_T|\alpha,A_m)$

(3.9) $g(\epsilon_t|\xi,v) = \begin{cases} \frac{2}{\xi+\frac{1}{\xi}}\varrho f[\xi(\varrho\epsilon_t+\bar{\omega})|v] & \text{if } \epsilon_t < -\bar{\omega}/\varrho \\ \frac{2}{\xi+\frac{1}{\xi}}\varrho f[(\varrho\epsilon_t+\bar{\omega})/\xi|v] & \text{if } \epsilon_t \geq -\bar{\omega}/\varrho \end{cases}$

(p122) $\varpi = \frac{\Gamma[(v-1)/2]\sqrt{v-2}}{\sqrt{\pi}\Gamma(v/2)}\left(\xi - \frac{1}{\xi}\right), \quad \varrho^2 = \left(\xi^2 + \frac{1}{\xi^2} - 1\right) - \varpi^2$

(3.10) $f(x) = \frac{v\exp\left(-\frac{1}{2}|x/\lambda|^v\right)}{\lambda 2^{(1+1/v)}\Gamma(1/v)}, \; -\infty < x < \infty, \; 0 < v \leq \infty$

(3.11) $\sigma_h^2(\ell) = \alpha_0 + \sum_{i=1}^{m}\alpha_i\sigma_h^2(\ell-i).$ where $\sigma_h^2(\ell-i) = a_{h+\ell-i}^2$ if $\ell - i \leq 0$

(3.14) $GARCH(m,s): a_t = \sigma_t\epsilon_t, \; \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m}\alpha_i a_{t-i}^2 + \sum_{j=1}^{s}\beta_j\sigma_{t-j}^2$

(3.15) $GARCH(m,s): a_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)}(\alpha_i+\beta_i)a_{t-i}^2 + \eta_t - \sum_{j=1}^{s}\beta_j\eta_{t-j}$

(p132) $\frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3[1-(\alpha_1+\beta_1)^2]}{1-(\alpha_1+\beta_1)^2-2\alpha_1^2} > 3$

(3.17) $\sigma_h^2(\ell) = \alpha_0 + (\alpha_1+\beta_1)\sigma_h^2(\ell-1), \quad \ell > 1$

(p133) $\sigma_h^2(\ell) = \frac{\alpha_0[1-(\alpha_1+\beta_1)^{\ell-1}]}{1-\alpha_1-\beta_1} + (\alpha_1+\beta_1)^{\ell-1}\sigma_h^2(1)$

(3.22) $\sigma_h^2(\ell) = \sigma_h^2(1) + (\ell-1)\alpha_0, \quad \ell \geq 1$

(3.23) $GARCH(1,1)-M: \; r_t = \mu + c\sigma_t^2 + a_t, \; a_t = \sigma_t\epsilon_t, \; \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1\sigma_{t-1}^2$

(3.24) $g(\epsilon_t) = \theta\epsilon_t + \gamma[|\epsilon_t| - E(|\epsilon_t|)]$

(p143) $g(\epsilon_t) = \begin{cases} (\theta+\gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t \geq 0 \\ (\theta-\gamma)\epsilon_t - \gamma E(|\epsilon_t|) & \text{if } \epsilon_t < 0 \end{cases}$

(p143) $E(|\epsilon_t|) = \frac{2\sqrt{v-2}\Gamma[(v+1)/2]}{(v-1)\Gamma(v/2)\sqrt{\pi}}$

(3.25) $EGARCH(m,s): a_t = \sigma_t\epsilon_t, \quad \ln(\sigma_t^2) = \alpha_0 + \frac{1+\beta_1 B+\cdots+\beta_{s-1}B^{s-1}}{1-\alpha_1 B-\cdots-\alpha_m B^m}g(\epsilon_{t-1})$

(3.26) $a_t = \sigma_t\epsilon_t, \quad (1-\alpha B)\ln(\sigma_t^2) = (1-\alpha)\alpha_0 + g(\epsilon_{t-1})$

(3.27) $$(1-\alpha B)\ln(\sigma_t^2) = \begin{cases} \alpha_* + (\gamma+\theta)\epsilon_{t-1} & \text{if } \epsilon_{t-1} \geq 0 \\ \alpha_* + (\gamma-\theta)(-\epsilon_{t=1}) & \text{if } \epsilon_{t-1} < 0 \end{cases}$$

(p144) $$\sigma_t^2 = \sigma_{t-1}^{2\alpha}\exp(\alpha_*) \begin{cases} \exp\left[(\gamma+\theta)\dfrac{a_{t-1}}{\sigma_{t-1}}\right] & \text{if } a_{t-1} \geq 0 \\ \exp\left[(\gamma-\theta)\dfrac{|a_{t-1}|}{\sigma_{t-1}}\right] & \text{if } a_{t-1} < 0 \end{cases}$$

(p148) $$\sigma_{h+1}^2 = \sigma_h^{2\alpha_1}\exp[(1-\alpha_1)\alpha_0]\exp[g(\epsilon_{h+1})]$$

(p148) $$E\{\exp[g(\epsilon)]\} = \exp\left(-\gamma\sqrt{2/\pi}\right)\left[e^{(\theta+\gamma)^2/2}\Phi(\theta+\gamma) + e^{(\theta-\gamma)^2/2}\Phi(\gamma-\theta)\right]$$

(p148) $$\hat{\sigma}_h^2(j) = \hat{\sigma}_h^{2\alpha_1}(j-1)\exp(\omega)\left\{\exp[(\theta+\gamma)^2/2\Phi(\theta+\gamma) + \exp[(\theta-\gamma)^2/2\Phi(\gamma-\theta)\right\}$$

Frequently Asked Questions in Quantative Finance, P. Wilmott

Q23 - Jensen's Inequality (103-105)

If the payoff is a convex function, $E[P(S_T)] \geq P(E[S_T])$

$$\begin{aligned} E[f(S)] = E\left[f(\bar{S}+\epsilon)\right] &= E\left[f(\bar{S}) + \epsilon f'(\bar{S}) + \frac{1}{2}\epsilon^2 f''(\bar{S}) + \cdots\right] \\ &\approx f(\bar{S}) + \frac{1}{2}f''(\bar{S})E[\epsilon^2] \\ &= f(E[S]) + \frac{1}{2}f''(E[S])E[\epsilon^2] \end{aligned}$$

The LHS is greater than the RHS by approximately $\frac{1}{2}f''(E[S])E[\epsilon^2]$

Q26 - Girsanov's Theorem (113-115)

The Theorem is:

Let W_t be a Brownian motion with measure P and sample space Ω. If γ_t is a previsible process satisfying the constraint $E_P\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2\right)\right] < \infty$ then there exists an equivalent measure Q on Ω such that $\tilde{W}_t = W_t + \int_0^t \gamma_s ds$ is a Brownian motion.

Chapter 6 - The Most Popular Probability Distributions

Normal or Guassian:

$$f(x) = \frac{1}{\sqrt{2\pi}b}\exp\left(-\frac{(x-a)^2}{2b^2}\right), -\infty < x < \infty, \mu = a, \sigma^2 = b^2$$

Lognormal:

$$f(x) = \frac{1}{\sqrt{2\pi}bx}\exp\left(-\frac{1}{2b^2}(\ln(x)-a)^2\right), x > 0, \mu = \exp\left(a + \frac{1}{2}b^2\right), \sigma^2 = \exp\left(2a+b^2\right)\left(\exp(b^2)-1\right)$$

Poisson:

$$p(x) = \frac{e^{-a}a^x}{x!}, x = 0, 1, \ldots, \mu = a, \sigma^2 = a$$

Chi square (note that this is a generalization of the usual chi-square distribution):

$$f(x) = \frac{e^{-(x+a)/2}}{2^{v/2}}\sum_{i=0}^{\infty}\frac{x^{i-1+v/2}a^i}{2^{2i}j!\Gamma(i+v/2)}, x \geq 0, \mu = v + a, \sigma^2 = 2(v+2a)$$

Gumbel:

$$f(x) = \frac{1}{b} \exp\left(\frac{a-x}{b}\right) \exp\left(-e^{\frac{a-x}{b}}\right), -\infty < x < \infty, \mu = a + \gamma b, \sigma^2 = \frac{1}{6}\pi^2 b^2 where \gamma = 0.577216\ldots$$

Weibull:

$$f(x) = \frac{c}{b}\left(\frac{x-a}{b}\right)^{c-1} \exp\left(-\left(\frac{x-a}{b}\right)^c\right), x > a, \mu = a + b\Gamma\left(\frac{c+1}{c}\right), \sigma^2 = b^2\left(\Gamma\left(\frac{c+2}{c}\right) - \Gamma\left(\frac{c+1}{c}\right)^2\right)$$

Student's t:

$$f(x) = \frac{\Gamma\left(\frac{c+1}{2}\right)}{b\sqrt{\pi c}\,\Gamma\left(\frac{c}{2}\right)}\left(1 + \frac{\left(\frac{x-a}{b}\right)^2}{c}\right)^{-\frac{c+1}{2}}, \ -\infty < x < \infty, \ \mu = a \text{ for } c > 1, \ \sigma^2 = \left(\frac{c}{c-2}\right) b^2 \text{ for } c > 2.$$

Pareto:

$$f(x) = \frac{ba^b}{x^{b+1}}, \ x \geq a, \ \mu = \frac{ab}{a-1} \text{ for } b > 1, \ \sigma^2 = \frac{a^2 b}{(b-2)(b-1)^2} \text{ for } b > 2$$

Uniform:

$$f(x) = \frac{1}{b-a}, \ a < x < b, \ \mu = \frac{a+b}{2}, \ \sigma^2 = \frac{(b-a)^2}{12}$$

Inverse normal:

$$f(x) = \sqrt{\frac{b}{2\pi x^3}} \exp\left(-\frac{b}{2x}\left(\frac{x-a}{a}\right)^2\right), \ x > 0, \ \mu = a, \ \sigma^2 = \frac{a^3}{b}$$

Gamma:

$$f(x) = \frac{1}{b\Gamma(c)}\left(\frac{x-a}{b}\right)^{c-1} \exp\left(\frac{a-x}{b}\right), \ x \geq a, \ \mu = a + bc, \ \sigma^2 = b^2 c$$

Logistic

$$f(x) = \frac{1}{b}\frac{\exp\left(\frac{x-a}{b}\right)}{\left(1 + \exp\left(\frac{x-a}{b}\right)\right)^2}, \ -\infty < x < \infty, \ \mu = a, \ \sigma^2 = \frac{1}{3}\pi^2 b^2$$

Laplace:

$$f(x) = \frac{1}{2b} \exp\left(-\frac{|x-a|}{b}\right), \ -\infty < x < \infty, \ \mu = a, \ \sigma^2 = 2b^2$$

Cauchy:

$$f(x) = \frac{1}{\pi b\left(1 + \left(\frac{x-a}{b}\right)^2\right)}, \ -\infty < x < \infty, \text{ moments do not exist}$$

Beta:

$$f(x) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)(b-a)^{c+d-1}}(x-a)^{c-1}(b-x)^{d-1}, \ a \leq x \leq b, \ \mu = \frac{ad+bc}{c+d}, \ \sigma^2 = \frac{cd(b-a)^2}{(c+d+1)(c+d)^2}$$

Exponential:

$$f(x) = \frac{1}{b} \exp\left(\frac{a-x}{b}\right), \ x \geq a, \ \mu = a + b, \ \sigma^2 = b^2$$

Lévy:

no closed form for density function, $\mu = \mu$, variance is infinite unless $\alpha = 2$, when it is $\sigma^2 = 2v$